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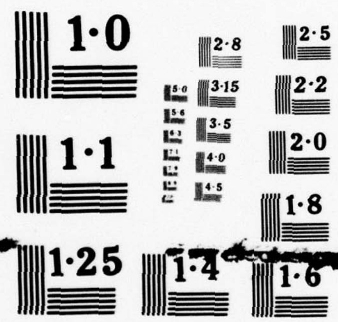

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Frequency Domain Stability Theory in Single and Multi-Dimensional Systems

by

R. Saeks, J. Murray, R. DeCarlo,
K. S. Chao, and E. C. Huang

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This report summarizes research into stability theory for single and multi-dimensional systems using frequency domain techniques. Specific items deal with the Nyquist test for multi-dimensional digital filters, spectral factorization for multi-dimensional digital filters, Half-plane filters, an algebraic-topological derivation of the classical Nyquist criterion, and a Continuation Algorithm for implementing the Multivariable Nyquist criterion.

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FREQUENCY DOMAIN STABILITY THEORY IN
SINGLE AND MULTIDIMENSIONAL SYSTEMS*

R. Saeks, J. Murray,
R.A. DeCarlo, K.S. Chao, and E.C. Huang

1978

*This research supported in part by AFOSR grant 74-2631.

CONTENTS

MULTIVARIABLE NYQUIST THEORY*

R.A. DeCarlo, J. Murray and R. Saeks

*This research supported in part by AFOSR grant 74-2631.

INTRODUCTION

Although the Nyquist criterion [1] has been known for over half a century, it has resisted generalization until recently. Interestingly, those generalizations which have been formulated retain the simple graphical character of the classical test, even when one is studying systems defined on abstract spaces. The earliest generalizations of the Nyquist criterion were the Circle and Popov criteria formulated in the early sixties as nonlinear and/or time-variable perturbations of the classical test [2], [3]. More recently MacFarlane [4], and Barman and Katznelson [5] have extended the test to the case of frequency response matrices while one of the authors has formulated a Nyquist-like sufficiency condition for Lipschitz continuous operators on abstract spaces [6]. Finally, in a recent paper the authors gave a stability test for multivariable digital filters which was formulated in terms of a continuum of Nyquist plots [7], [8]. In all cases the tests remain simple graphical conditions on the complex plane. The resultant criteria yield necessary and sufficient conditions in the case of linear time-invariant systems (including the multivariable and matrix generalizations) and sufficient conditions in the cases of nonlinear and time-variable systems.

The thrust of this paper is to show that with a slight modification, the continuum of Nyquist plots used in the multivariable Nyquist test of [7] can be reduced to a classical single variable Nyquist plot plus a test to verify that the filter has no poles in the region $|z_1| = |z_2|$

$= \ldots = |z_n| = 1$, the multivariable analog of the $i\omega$- axis.

Although every attempt is made to minimize details the theorem illustrates the essential algebraic topological nature of the Nyquist criterion, with homotopic arguments playing a significant role in derivation. For a more detailed discussion of the algebraic topological nature of Nyquist theory the reader is refered to [7], [8] and [9]. In particular, [9] gives an algebraic topological derivation of the classical Nyquist criterion.

In the following section several Hurwitz-like stability tests are reviewed and a new test of the Hurwitz-type test is formulated. In the third section a homotopic interpretation of the classical Nyquist criterion is formulated, this being used to derive the desired multivariable Nyquist criterion from the Hurwitz conditions in the fourth and fifth sections. Finally some examples of the theory are given in section 6.

Hurwitz-like Tests

Denote the vector space of complex n-tuples by $\mathbb{C}^n$. For the purposes of our multivariable stability theory, there are five interesting subsets of $\mathbb{C}^n$. First there is the polydisk, defined as

$$P^n = \{(z_1,\ldots,z_n) \text{ in } \mathbb{C}^n \mid |z_i| \leq 1, \ i = 1,\ldots,n\} \tag{1}$$

It plays the same role in the multivariable theory as the unit disk (or right half plane) in single variable theory.

Next, there are three separate notions of the boundary of P^n. All are necessary for the theory of this paper. First is the dis-

<u>tinguished</u> <u>boundary</u> denoted by T^n where

$$T^n = \{(z_1,\ldots,z_n) \text{ in } \mathbb{C}^n \;\Big|\; |z_i| = 1,\ i = 1,\ldots,n\} \tag{2}$$

T^n serves as the multidimensional analog of the $i\omega$-axis. In particular the frequency response [10] of a digital filter is the evaluation of its transfer function over T^n. Geometrically T^n is an n-dimensional torus which reduces to the unit circle of the complex plane in the single variable case. T^n is a "boundary" for P^n in the sense that it is a subset of P^n for which all coordinants of P^n simultaneously take on extremal values.

A second notion of boundary for P^n is defined by requiring only that n-1 coordinats take on extremal values. This boundary set is denoted by M^n where

$$M^n = \{(z_1,\ldots,z_n) \text{ in } \mathbb{C}^n \;\Big|\; |z_i| = 1,\ i = 1,\ldots,k-1,k+1,\ldots,n;\ |z_k| \leq 1\} \tag{3}$$

and where k ranges from 1 through n.

The final notion of boundary requires that at least one of the coordinates take on extremal values. This notion is the usual <u>topological</u> <u>boundary</u> since it coincides with the usual concept of boundary of the set P^n in the sense of point set topology [11]. The topological boundary is denoted by B^n where

$$B^n = \{(z_1,\ldots,z_n) \text{ in } \mathbb{C}^n \;\Big|\; |z_i| \leq 1,\ i = 1,\ldots,n \text{ and } |z_k| = 1 \text{ for some } k\} \tag{4}$$

Finally we define a subset, H^n, of P^n whose relevence to the stability problem was originally indicated by Huang [12].

$$H^n = \{(z_1,\ldots,z_n) \text{ in } \mathcal{C}^n \mid |z_i| = 1,\ i\text{-}1,\ldots,k\text{-}1;\ |z_k| \leq 1;\ z_i = 0, i=k+1,\ldots,n\}; \quad (5)$$

Here k varies from 1 through n. Note, that $M^1 = H^1 = P^1$ and $T^1 = B^1$ hence these sets become redundant and all reduce to either the unit disk or unit circle in the single variable case.

Shanks [13] was the first to give a Hurwitz-like test for the stability of multidimensional digital filters. His condition states essentially that the filter transfer function must have no poles in P^n. In the single variable case, the pole set of a transfer function is discrete. However, in the case of higher dimensional filters, the pole set is an infinite continuum. Using this fact, Huang [12] showed that a transfer function has a pole in P^2 if and only if it also has a pole in H^2. This is not toimply that the only poles of the transfer function lie in H^2 but rather that the pole set is so large that it cannot pass through P^2 without intersecting the subset H^2. Anderson and Jury [14] extended Haung's theorem to the n-dimensional case by showing that a transfer function has a pole in P^n if and only if it has a pole in H^n. The proof of Haung's theorem and its generalization is tediously straightforward but requires a clever application of the maximum modulus theorem.

A result somewhat similar to Huang's can be formulated in terms of the topological boundary. To derive such a condition, one exploits the fact that the pole set of a multivariable ($n \geq 2$) rational function is an infinite continuum (more precisely no connected component of the pole set is compact [15]). As such, the only way the pole set can intersect P^n is if it crosses the topological boundary, B^n. This

implies that a transfer function has a pole set intersecting P^n if and only if the pole set has a non-void intersection with B^n.

Now observe that B^n can be viewed as the union of a family of (n-1) variable polydisks (parameterized by k and the value of z_k, $|z_k| = 1$). Hence the above argument can be repeted to show that the transfer function has a pole set intersecting the topological boundary of such an (n-1)-variable polydisk if it has a pole set intersection P^n. Upon iterating the argument (n-1) times and eliminating redundant sets, one eventually arrives at the following condition: the transfer function has a pole in P^n if and only if it has a pole in M^n.

The above various Hurwitz-like stability tests for multivariable digital filters are summarized as follows:

<u>Therorem 1</u>: Let a causal multidimensional digital filter be characterized by a rational transfer function in several complex variables. Assume the numerator and denominator polynomials are relatively prime. Then the following are equivalent stability conditions:

i) the pole set of the transfer function has a null intersection with P^n.

ii) the pole set of the transfer function has a null intersection with B^n.

iii) the pole set of the transfer function has a null intersection with H^n.

iv) the pole set of the transfer function has a null intersection with M^n.

The easiest way to evaluate the stability tests based on the

above conditions is by a comparison of the (topological) dimension of the sets where one checks for the existence of poles. In particular, P^n is 2n-dimensional, B^n is (2n-1)-dimensional, while H^n and M^n are both (n+1)-dimensional. Again realize that the equivalences of Theorem 1 follow from the fact that the pole set is an infinite continuum whose complex dimension is (m-1) where m is the number of complex variables of the specific function. Finally observe that a pole is implicitly used to mean a specific point in the "pole set."

Nyquist Theory

The task of this section is to construct the concepts of a Nyquist contour, a Nyquist plot, encirclement, and degree, all in topological terms which are thus extendable to the multivariable case. All of our conditions will be stated in terms of the zero set of a relatively prime denominator polynomial of a transfer function. Hence we will deal exclusively with polynomials in several complex variables rather than rational transfer functions.

Traditionally engineers view the Nyquist contour as a subset of the complex plane. This point of view is somewhat erroneous. Mathematically speaking the Nyquist contour (the usual closed semicircle, the imaginary axis, or the unit circle) is a continuous map (of bounded variation) from T^1 to $\mathbb{C}^n$. The image of this map, called the trace of the map, is the traditional engineering notion of the Nyquist contour. In this paper T^1 is the unit circle of the complex plane. In the single variable case (classical digital filter stability) one works with a

"Nyquist contour" defined by $\Gamma(\alpha) = \alpha = \exp(i\theta)$ for $0 \leq \theta < 2\pi$, $\theta = \arg(\alpha)$. In the multivariable case, the map will take on values in $\mathbb{C}^n$ forcing the "Nyquist contour" to be a more involved entity. Observe that we are taking liberties with the calssical definition of the Nyquist contour and plot. In an abstract sense, there is no essential difference although the specific applications (classical feedback stability or presently digital filter stability) are somewhat alien.

In this paper a <u>Nyquist plot</u> is defined as the composition of the Nyquist contour, Γ, with a polynomial in several complex variables, Γ, as per Figure 1. Note that

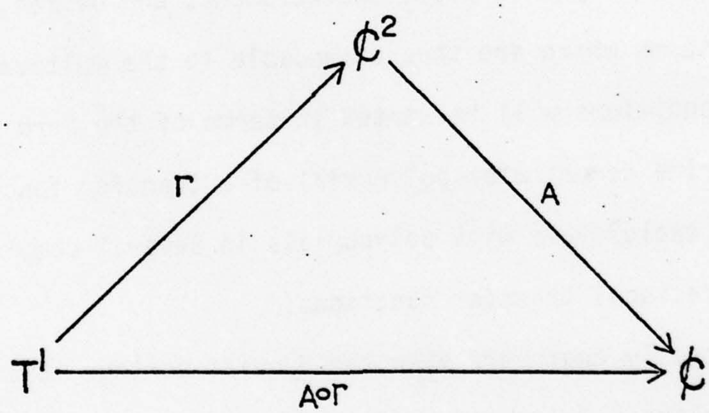

Figure 1: The Nyquist plot as a composition of maps.

the polynomial map from $\mathbb{C}^n$ to $\mathbb{C}$ is analytic and it is this property which makes the theory go. Thus the Nyquist plot is a continuous map of bounded variation from T^1 to $\mathbb{C}$.

The concept of encirclement is intimately related to the topological concept of homotopy. Keeping this association in mind, let X be an arbitrary topological space and let μ and γ be continuous functions

of bounded variation defined on T^1 with values in X: $\mu:T^1 \to X$ and $\lambda:T^1 \to X$. The maps μ and λ are said to be homotopic if there exists a continuous map, $\emptyset$, defined on the product space $T^1 \times I$, $I = [0,1]$, with values in X such that $\emptyset(\alpha,0) = \mu(\alpha)$ and $\emptyset(\alpha,1) = \lambda(\alpha)$. In essence, a homotopy is a continuous deformation of the curve μ into the curve λ. This concept defines an equivalence relation on the set of continuous maps from T^1 to X--i.e. two curves are equivalent if one can be continuously deformed into the other. A curve is said to be homotopically trivial if it is homotopic to a constant map. Note that the use of an abstract topological space, X, in the definition of homotopy, is fundamental to the concept, since all curves with values in $\mathbb{C}^n$ or R^n are homotopically trivial. Although we are interested in the properties of functions defined on $\mathbb{C}^n$, a number of non-trivial topological spaces arise in our analysis. In particular the torus, T^n, and the punctured plane. Here the concept of encirclement may be defined for maps taking their values in $\mathbb{C} - \{0\}$. This is a highly nontrivial space in which the distinct equivalence classes of homotopic maps can be indexed by the integers corresponding to the number of times a curve encircles the point zero. This number is termed the degree, $n(\gamma,0)$, of the map, γ, and can be computed by the formula [11]

$$n(\gamma,0) = \frac{1}{2\pi i} \int_{\gamma} (z-0)^{-1} dz \tag{6}$$

where γ is the curve in question and γ does not take on the value zero. The desire is to make a binary decision on whether or not the map encircles zero. This may be defined in purely homotopic terms by saying

that a map $\mu:T^1 \to \mathbb{C} - \{0\}$ does not encircle zero if it is homotopically trivial (homotopic to a constant map). The concept may then be extended to maps taking values in $\mathbb{C}$ which do not pass through the point zero by viewing them as maps with values in $\mathbb{C} - \{0\}$.

The significance of these homotopic concepts in algebraic topology [11] is due to the fact that the equivalence classes of homotopic maps form a group, $\pi(X)$, where X is the space in question. $\pi(X)$ is called the fundamental group of X. More precisely, if we have two maps $\mu:T^1 \to X$ and $\lambda:T^1 \to X$, such that $\mu(1) = \lambda(1)$, their concatenation $\mu * \lambda:T^1 \to X$ is defined by

$$[\mu*\lambda](\alpha) = \begin{cases} \lambda(\alpha^2) & 0 \leq \arg(\alpha) \leq \pi \\ \mu(\alpha^2) & \pi \leq \arg(\alpha) < 2\pi \end{cases}$$

Intuitively, $\mu*\lambda$, is a curve which first follows λ as $\arg(\alpha)$ goes from 0 to π and then follows μ as $\arg(\alpha)$ goes from π to 2π. Moreover, since $\mu(1) = \lambda(1)$, then $\mu*\lambda$ is continuous if μ and λ are continuous. Clearly concatenation is invariant under homotopic equivalence [11]. (i.e. if μ_1 is homotopic to μ_2 and λ_1 is homotopic to λ_2 then $\mu_1 * \lambda_1$ is homotopic to $\mu_2 * \lambda_2$). As such it defines a binary operation on the equivalence classes of homotopic curves $\mu:T^1 \to X$ with a fixed value for $\mu(1)$. Thus the group operation of $\pi(X)$ is concatenation [11]. In classical algebraic topology, the properties of $\pi(\mathbb{C}^n) = 0$ since all maps taking values in $\mathbb{C}^n$ are homotopically trivial. On the other hand $\pi(\mathbb{C} - \{0\})$ is isomorphic to the additive group of integers where the "degree function" is the isomorphism.

Using this machinery we now formalize a statement of the "classical" Nyquist criterion. Consider the "obvious" Nyquist contour, $\Gamma:T^1 \to \mathbb{C}$ defined as $\Gamma(\alpha) = \alpha = \exp[i \arg(\alpha)]$ with the corresponding Nyquist plot $A \circ \Gamma : T^1 \to \mathbb{C}$ where A is a polynomial in one variable.

Theorem 2: (Nyquist) Let A be a polynomial on $\mathbb{C}$. Then A has no zeros in P^1 (the unit closed disk) if and only if $A \circ \Gamma$ does not pass through nor encircle zero.

Main Theorems

Here the multivariable Nyquist theory is derived from condition (iv) of Theorem 1. Let a causal digital filter transfer function be $H(z_1,\ldots,z_n) = B(z_1,\ldots,z_n)/A(z_1,\ldots,z_n)$ where A and B are relatively prime. The system so characterized is stable if and only if A has no zeros in M^n--i.e. the zero set of A does not intersect M^n. Now M^n can be expressed as a union of single variable polydisks as follows. First, for any given set of (n-1) elements, α_i, of T^1, indexed by the integers $1,2,\ldots,k-1, k+1,\ldots,n$, embed a single variable polydisk into $\mathbb{C}^n$ as

$$\begin{aligned} &P^1(\alpha_1,\ldots,\alpha_{k-1},\cdot,\alpha_{k+1},\ldots,\alpha_n) \\ &\quad = \{(z_1,\ldots,z_n) \text{ in } \mathbb{C}^n \mid z_i = \alpha_i,\ i = 1,\ldots,k-1, \\ &\qquad k+1,\ldots,n; |z_k| \leq 1\} \end{aligned} \tag{8}$$

By comparison with equation 3, one can verify that

$$M^n = \bigcup_{k=1}^{n} \ \bigcup_{\alpha_1 \text{ in } T^1} P^1(\alpha_1,\ldots,\alpha_{k-1},\cdot,\alpha_{k+1},\ldots,\alpha_n)$$

Hence the digital filter is stable if and only if A has no zeros in

each of the polydisks $P^1(\alpha_1,\ldots,\alpha_{k-1},z_k,\alpha_{k+1},\ldots,\alpha_n)$. Moreover, since these are dependent on only one coordinant, one may test for zeros of A in $P^1(\alpha_1,\ldots,\alpha_{k-1},\cdot,\alpha_{k+1},\ldots,\alpha_n)$ by sequentially testing for zeros of the single variable polynomial $A(\alpha_1,\ldots,\alpha_{k-1},z_k,\alpha_{k+1},\ldots,\alpha_n)$ in the single variable polydisk P^1 as defined in the introduction. Each such test can be executed using the "Nyquist theorem." This yields a stability test for a function of several variables which takes the form of a continuum of classical Nyquist plots.

Lemma 3: Let A be a polynomial mapping $\mathbb{C}^n$ to $\mathbb{C}^n$. Then A has no zeros in P^n if and only if each of the Nyquist plots for the family of single variable polynomials.

$A(\alpha_1,\ldots,\alpha_{k-1},z_k,\alpha_{k+1},\ldots,\alpha_n)$ for $k+1,\ldots,n$ and α_i in T^1, do not pass through nor encircle zero.

Lemma 3 is essentially equivalent to the condition formulated in references [7] and [8]. However, the present condition arose from the fact that A has no zeros in M^n whereas the previous test grew from the fact that A has no zeros in H^n. As was shown in references [7] and [8] the lemma can be implemented as a practical test in the two dimensional case. Here one simply chooses a finite set of α's in T^1 and plots the corresponding Nyquist loci. Since T^1 is compact, this discretization can be made to yield as much accuracy as desired. Unfortunately, in the multivariable case our family of Nyquist plots is parameterized by "n" (n-1)-dimensional tori.

The purpose of the first main theorem is to show that the family of Nyquist plots of Lemma 3 is reducible to "n" classical single variable

Nyquist plots and one further test. To this end we reformulate the classical Nyquist contour used to test the single variable function $A(\alpha_1,\ldots,\alpha_{k-1},z_k,\alpha_{k+1},\ldots,\alpha_n)$ as a Nyquist contour taking its values in $\mathbb{C}^n$; define the Nyquist contour, $\Gamma(\alpha_1,\ldots,\alpha_{k-1},\cdot,\alpha_{k+1},\ldots,\alpha_n):T^1\to\mathbb{C}^n$ by the equality

$$\Gamma(\alpha_1,\ldots,\alpha_{k-1},\alpha,\alpha_{k+1},\ldots,\alpha_n) = (\alpha_1,\ldots,\alpha_{k-1},\alpha,\alpha_{k+1},\ldots,\alpha_n) \tag{9}$$

Clearly, the Nyquist plot $A\circ\Gamma(\alpha_1,\ldots,\alpha_{k-1},\alpha,\alpha_{k+1},\ldots,\alpha_n)$ coincides with the Nyquists plots of Lemma 3. Thus Lemma 3 can be reformulated in terms of these plots. The key attribute of these multivariable Nyquist contours is that the entire family of Nyquist contours for a fixed k are homotopically equivalent.

Lemma 4: For any given set of α_i's in T^1, $i = 1,\ldots,k-1,k+1,\ldots,n$, the Nyquist contour $\Gamma(\alpha_1,\ldots,\alpha_{k-1},\cdot,\alpha_{k+1},\ldots,\alpha_n)$ is homotopic in T^n to the Nyquist contour $\Gamma(1,\ldots,1,\cdot,1,\ldots,1)$.

Proof: Consider the homotopy $\emptyset:T^1\times I\to T^n$ defined by

$$\emptyset(\alpha,t) = (\exp[i\theta_1(1-t)],\ldots,\exp[i\theta_{k-1}(1-t)],\alpha,\exp[i\theta_{k+1}(1-t)],\ldots,\exp[i\theta_n(1-t)]) \tag{10}$$

where $\theta_i = \arg(\alpha_i)$ for any set $\lambda_i\ \varepsilon T^1$, $i = 1,\ldots,k-1,k+1,\ldots,n$. Here $\emptyset(\alpha,t)$ is in T^n for all α and t. Moreover

$$\emptyset(\alpha,0) = [\exp(i\ \theta_1),\ldots,\exp(i\theta_{k-1}),\alpha,\exp(i\theta_{k+1}),\ldots,\exp(i\theta_n)] = \Gamma(\alpha_1,\ldots,\alpha_{k-1},\alpha,\alpha_{k+1},\ldots,\alpha_n) \tag{11}$$

and

$$\emptyset(\alpha,1) = (1,\ldots,1,\alpha,1,\ldots,1) = \Gamma(1,\ldots,1,\alpha,1,\ldots,1) \tag{12}$$

Hence $\emptyset$ is the desired homotopy.

Theorem 3: Let A be a polynomial on $\mathbb{C}^n$. Then A has no zeros in P^n if and only if:

(i) A has no zeros on T^n, and

(ii) The Nyquist plots for the single variable functions $A(1,\ldots,1,z_k,1,\ldots,1)$ $k = 1,\ldots,n$ do not encircle zero.

Proof: Since the images of the family of Nyquist contours defined by equation 9 cover T^n, the fact that none of the Nyquist plots of Lemma 3 go through zero implies that A has no zeros on T^n. This verifies the necessity of condition (i). To verify the necessity of condition (ii) observe that the "n" Nyquist plots of condition (ii) are a subset of the family of Nyquist plots of Lemma 3.

To verify the sufficiency of the theorem, observe that, if A has no zeros on T^n, then A restricted to T^n is a continuous map from T^n to $\mathbb{C} - \{0\}$. Now, since the continuous images of homotopic maps are homotopic, the fact that the Nyquist contours $\Gamma(\alpha_1,\ldots,\alpha_{k-1},\cdot,\alpha_{k+1},\ldots,\alpha_n)$ and $\Gamma(1,\ldots,1,\cdot,1,\ldots,1)$ are homotopic for any fixed k with α_i in T^1, $i = 1,\ldots,k-1,k+1,\ldots,n$, implies that the Nyquist plots $A\circ\Gamma(\alpha_{k-1},\cdot,\alpha_{k+1},\ldots,\alpha_n)$ and $A\circ\Gamma(1,\ldots,1,\cdot,1,\ldots,1)$ are homotopic in $\mathbb{C} - \{0\}$; hence all such Nyquist plots encircle zero if and only if the Nyquist plot for $A\circ\Gamma(1,\ldots,1,\cdot,1,\ldots,1)$ encircles. As such, if A has no zeros on T^n we are assured that none of the Nyquist plots of Lemma 3 go through zero, whereas if the "n" Nyquist plots

$A_{or}(1,\ldots,1,\cdot,1,\ldots,1)$ do not encircle zero, then none of the Nyquist plots of Lemma 3 encircle zero. With the final observation that the Nyquist plot for the single variable function $A(1,\ldots,1,z_n,1,\ldots,1)$ coincides with $A_{or}(1,\ldots,1,\cdot,1,\ldots,1)$ this verifies the sufficiency of the theorem.

Surely this theorem is a true generalization of the classical Nyquist theorem in that it tests for stability, using only distinguished boundary ($i\omega$-axis) information. Moreover, the test is n-dimensional, hence superior to any of the Hurwitz-type tests.

Intuitively speaking the result is both surprising and expected. It is surprising because one tests for zeros of an n-variable function using single variable Nyquist plots as opposed to some type of n-dimensional encirclement. It was expected, however, since a polynomial contains a finite amount of information (a finite number of coefficients), so that only a finite number of tests need be executed. In this light the condition of Lemma 3 seemed superfluous.

Again, Theorem 3 is aesthetically pleasing since it uses only frequency response information. However, by cleverly considering the implications of this information as per [12], [14], one may concoct an equivalent test. Essentially the test will be a consequence of condition (iii) of Theorem 1 and hopefully will be easier to implement.

Theorem 4: Let A be a polynomial mapping $\mathbb{C}^n$ to $\mathbb{C}^n$. Then A has no zeros in P^n if and only if

(i) A has no zeros on T^n, and

(ii) The Nyquist plots for the single variable functions $A(1,\ldots,1,z_k,0,\ldots,0)$ $k = 1,\ldots,n$, do not encircle zero.

<u>Proof</u>: Note first, that from [7], and [8], it is known that A has no zeros in P^n if and only if the Nyquist plots of $A(\alpha_1,\ldots,\alpha_{k-1},z_k,0,\ldots,0)$ for $k = 1,\ldots,n$ and for each and every α_i in T^1 do not encircle nor pass through zero.

Since A has no zeros in P^n, it immediately follows that A has no zeros on T^n. Moreover, the Nyquist plots of condition (ii) are a subset of the Nyquist plots of the stability result noted above. Thus the forward direction is shown.

Conversely suppose condition (i) and condition (ii) hold. Since A has no zeros in T^n, Lemma 4 guarantees that the Nyquist plots of $A(\alpha_1,\ldots,\alpha_{n-1},z_n)$ for each and every α_i in T^1 are homotopic to one another. In particular they are homotopic to $A(1,\cdots,1,z_n)$. Thus any member of this family of Nyquist plots encircles zero if and only if $A(1,\ldots,1,z_n)$ encircles zero.

Now if the Nyquist plot of $A(1,\ldots,1,z_n)$ does not encircle zero, then $A(\alpha_1,\ldots,\alpha_{n-1},0) \neq 0$ for each and every α_i in T^1. In other words $A(z_1,\ldots,z_{n-1},0)$ has no zeros on the n-1 dimensionaltorus; $\{(z_1,\ldots,z_n) \in C^n \,\big|\, |z_i| = 1; i = 1,\ldots,n-1; z_n = 0\}$. Repeating the above arguments we conclude that the Nyquist plot of $A(1,\ldots,1,z_{n-1},0)$ encircles zero if and only if each of the Nyquist plots of $A(\alpha_1,\ldots,\alpha_{n-2},\alpha_{n-1},0)$ encircles zero. Continuing in this fashion, one verifies that the conditions of the theorem are equivalent to the condition of references [7], [8] as stated at the beginning of the proof. The statement of the theorem now follows.

Observe that this theorem does not explicitly use distinguished

boundary (frequency response) information. However, the sequential manner of the test does use such information in an implicit way. Also the test of Theorem 4 may be easier to implement since the zero dependence of $A(1,\dots,z_k,0,\dots,0)$ tends to cancel terms in the original polynomial which effectively diminishes the complexity of the test.

At this point we state and prove the final theorem of the section. This theorem combines the "n" Nyquist plots of Theorem 3 into a single Nyquist plot.

Theorem 5: Let A be a polynomial mapping $\mathbb{C}^n$ into $\mathbb{C}^n$. Then A has no zeros in P^n if and only if

(i) A has no zeros on T^n, and

(ii) The Nyquist plot for the single variable function $A(z,z,\dots,z)$ does not encircle zero.

Before giving the formal proof of this theorem, we will sketch its derivation as a corollary to Theorem 3. First observe that since an analytic function is orientation preserving, [15], the degree of each of the single variable Nyquist plots, $A\circ\Gamma(1,1,\dots,1,\cdot,1,\dots,1)$ of Theorem 3, is non-negative. Moreover, the degree function associated with the set of closed curves is an isomorphism from $\pi(C - \{0\})$ onto the additive group of integers. Thus the degree of the concatenation (the group operation in $\pi(C - \{0\})$) of closed curves, will be the sum of the degrees of the individual curves. Furthermore, the sum of non-negative integers is zero if and only if each integer is zero. This implies that the second hypothesis of Theorem 3 holds if and only if the single Nyquist plot, obtained by concatenating the "n" Nyquist plots of Theorem 3,

does not encircle zero. Consequently Theorem 3 reduces to verifying that A has no zeros on T^n and checking their encirclement of zero by the single Nyquist plot:

$$[A\circ\Gamma(\cdot,1,\ldots,1)]*[A\circ\Gamma(1,\cdot,1,\ldots,1)]* \ldots *[A\circ\Gamma(1,\ldots,1,\cdot)]$$

$$= A\circ[\Gamma(\cdot,1,\ldots,1)*\Gamma(1,\cdot,1,\ldots,1)* \ldots *\Gamma(1,\ldots,1,\cdot)] \qquad (13)$$

where the equality of equation [13] is due to the fact [11] that composition distributes over concatenation.

Although equation 13 reduces the multivariable stability test to a single Nyquist plot, this plot is just the concatenation of the n plots of Theorem 3. The resultant test, then, is no easier to implement than the original test. Fortunately, this Nyquist plot is homotopic in $C - \{0\}$ to the Nyquist plot for the single variable function $A(z,z,\ldots,z)$, obtained by setting each of the dependent variables of A equal to one another. To verify this contension, first observe that this single variable Nyquist plot is equal to the Nyquist plot $A\circ\Gamma(\cdot,\cdot,\ldots,\cdot)$ where $\Gamma(\cdot,\cdot,\ldots,\cdot):T^1 \text{---} C^n$ by taking the point α in T^1 to $(\alpha,\alpha,\ldots,\alpha)$ in C^n. Now, A has no zeros on T^n. Hence a maps T^n continuously to $C - \{0\}$. The Nyquist plot $A\circ\Gamma(\cdot,\cdot,\ldots,\cdot)$ will thus be homotopic in $C - \{0\}$ to the Nyquist plot of equation 13 provided their corresponding Nyquist contours are homotopic in T^n. This is, indeed, the case. However, the required homotopy is extremely complex. As such, rather than wading through the details, we will simply sketch the required homotopy in the two variable case. Then we proceed to an alternate proof of the Theorem based on a known, but non-intuitive, theorem of functions of several complex variables.

To sketch the required homotopy in the two variable case, represent the torus, T^2, as a square on the plane. Topologically identify opposite sides of the squares. Figure 2 illustrate such a square. The point $(z_1,z_2), |z_1| = |z_2| = 1$, in T^2, corresponds to the point (θ_1,θ_2) on the plane where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$ i.e. $(z_1,z_2) = (e^{i\theta_1},e^{i\theta_2})$. In otherwords the upper and lower boundaries of the square represent the same line on the torus since $e^{i\pi} = e^{-i\pi}$ and similarly for the right and left boundaries of the square. Moreover, all four corners of the square represent the same point, (-1,-1). In the sketch of Figure 2, the Nyquist contour $\Gamma(\cdot,1)*\Gamma(1,\cdot)$ of equation [13] corresponds to the curve, number 1, which starts at $(\theta_1,\theta_2) = (0,0)$ in the center of the square traveling vertically to the top of the square. It then goes from the bottom of the square vertically back to the center, from the center of the square curve 1 then passes horizontally to the right hand boundary, and finally it returns from the left hand boundary of the square back to the center. Since the upper and lower boundaries of the square are identified, when the curve "jumps" from the upper to lower boundary, it remains continuous (think of the square being rolled up into a cylinder with the upper and lower boundaries glued together). Similarly for the "jump" from the right to left boundary. Of course, curve 1 is closed since it starts and ends at the same point.

The Nyquist contour $\Gamma(\cdot,\cdot)$ is represented in Figure 2 by curve 5, which starts at the center of the square, goes diagonally to the upper right hand corner of the square and then "jumps" to the lower left hand corner of the square from which point it returns diagonally back to the center. As before the curve is continuous and closed.

The required homotopy between curves 1 and 5 is indicated on Figure 2 by the three intermediary curves numbered 2, 3, and 4. As before these curves are continuous since the upper and lower boundaries and the left and right boundaries of the square are identified. Also all the intermediary curves begin and end at the base point $(\theta_1,\theta_2) = (0,0)$. The continuity of the intermediary curves is illustrated graphically in Figure 3 wherein we have redrawn Figure 2 with the point $(\theta_1,\theta_2) = (\pm\pi, \pm\pi)$ taken as the center point. In this representation it is clear that curves 2, 3, and 4 are continuous and converge to curve 5.

Although the homotopy required to complete our proof is neatly illustrated in Figure 2, its explicit mathematical discription is by no means simple, even for the two variable case. Consequent rather than formalizing the tedious details of the required n-variable homotopy we construct an alternate proof of Theorem 5 based on a theorem of several complex variables. Since the theorem is applicable to analytic functions as well as polynomials this proof will also allow us to extend theorem 5 to the case of meramorphic transfer functions in several complex variables.

<u>Lemma 5</u>: Let $f = (f_1,f_2,\ldots,f_n)$ be a continuous function mapping P^1 to $C^n, n \geq 2$, such that $f(T^1) \subset T^n$ and each of its coordinate functions, f_i, have positive degree when viewed as functions from T^1 to C, $f_i\big|_{T^1}:T^1\to C$. Then for any analytic function $g:C^n\to C$, g has a zero in P^n if and only if g has a zero in $T^n \cup f(P^1)$. The theorem appears on page 87 of reference 17 and its proof will not be repeated here. In essence the theorem yields an entire family of n-dimensional Hurwitz-like tests (since T^n is n-dimensional and $f(P^1)$ is 2-dimensional) one for each f satisfying the hypotheses

of the theorem.

Proof of Theorem 5: To prove theorem 5, we apply Lemma 5 with f defined by $f(z) = (z,z,\ldots,z)$. Since each coordinate function $f_i(z) = z$ is the identity map the coordinates all have degree one the hypotheses of Theorem 5 are satisfied. As such, the polynomial $A:C^n \to C$ (an arbitrary analytic function, $g:C^n \to C$, could be used with equal validity) has a zero in P^n if and only if it has a zero in $T^n \cup f(P^1)$. Now,

$$f(P^1) = \{(z,z,\ldots,z) \text{ in } C^n \,\big|\, |z| \leq 1\} \tag{14}$$

is just a polydisk in one variable embedded in C^n. As such, via the classical single variable Nyquist criterion the existence of zeros of A in $f(P^1)$ may be determined using the Nyquist plot $A\circ\Gamma(\cdot,\cdot,\ldots,\cdot)$ whose Nyquist contour $\Gamma(\cdot,\cdot,\ldots,\cdot)$ follows the boundary of $f(P^1)$. This Nyquist plot is, however, just the classical Nyquist plot for the single variable function $A(z,z,\ldots,z)$. Thus, if we check to see if A has no zeros on T^n and that the Nyquist plot for $A(z,z,\ldots,z)$ does not encircle zero we are assured that A has no zeros in $T^n \cup f(P^1)$ and thus by Theorem 6 we are assured that A has no zeros in P^n as was to be shown.

Examples

Example 1

Consider the six-variable fourth order polynomial

$$A(z_1,z_2,z_3,z_4,z_5,z_6) = 10z_1z_2z_3z_4z_5^2 + 2z_1^2z_2 + z_1z_3^2z_4^4 + z_6^3 + 3 \tag{15}$$

For which the image of A restricted to T^n is plotted in Figure 4. Since zero is not in the image condition A. of the theorem is satisfied and we may proceed to check condition B. This requires that we test the

Nyquist plots for the six one variable functions

$$A(z_1,1,1,1,1,1) = 11z_1 + 2z_1^2 + 4 \qquad (16.)$$

$$A(1,z_2,1,1,1,1) = 12z_2 + 5 \qquad (17.)$$

$$A(1,1,z_3,1,1,1) = 10z_3 + z_3^2 + 6 \qquad (18.)$$

$$A(1,1,1,z_4,1,1) = 10z_4 + z_4^4 + 6 \qquad (19.)$$

$$A(1,1,1,1,z_5,1) = 10z_5^2 + 7 \qquad (20.)$$

and

$$A(1,1,1,1,1,z_6) = z_6^3 + 16 \qquad (21.)$$

for encirclements of zero. The resultant plots are sketched in Figures 5a through 5f. where we see that five of the six plots encircle zero. As such, the system is unstable.

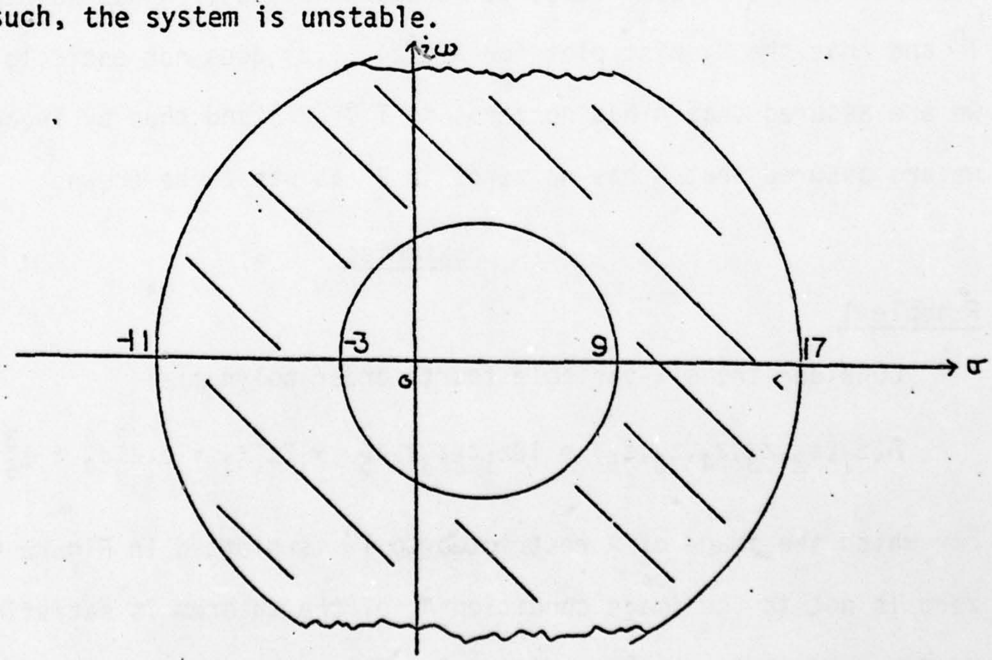

Figure 4. Plot of $A(T^n)$ for the six variable function of equation 12.

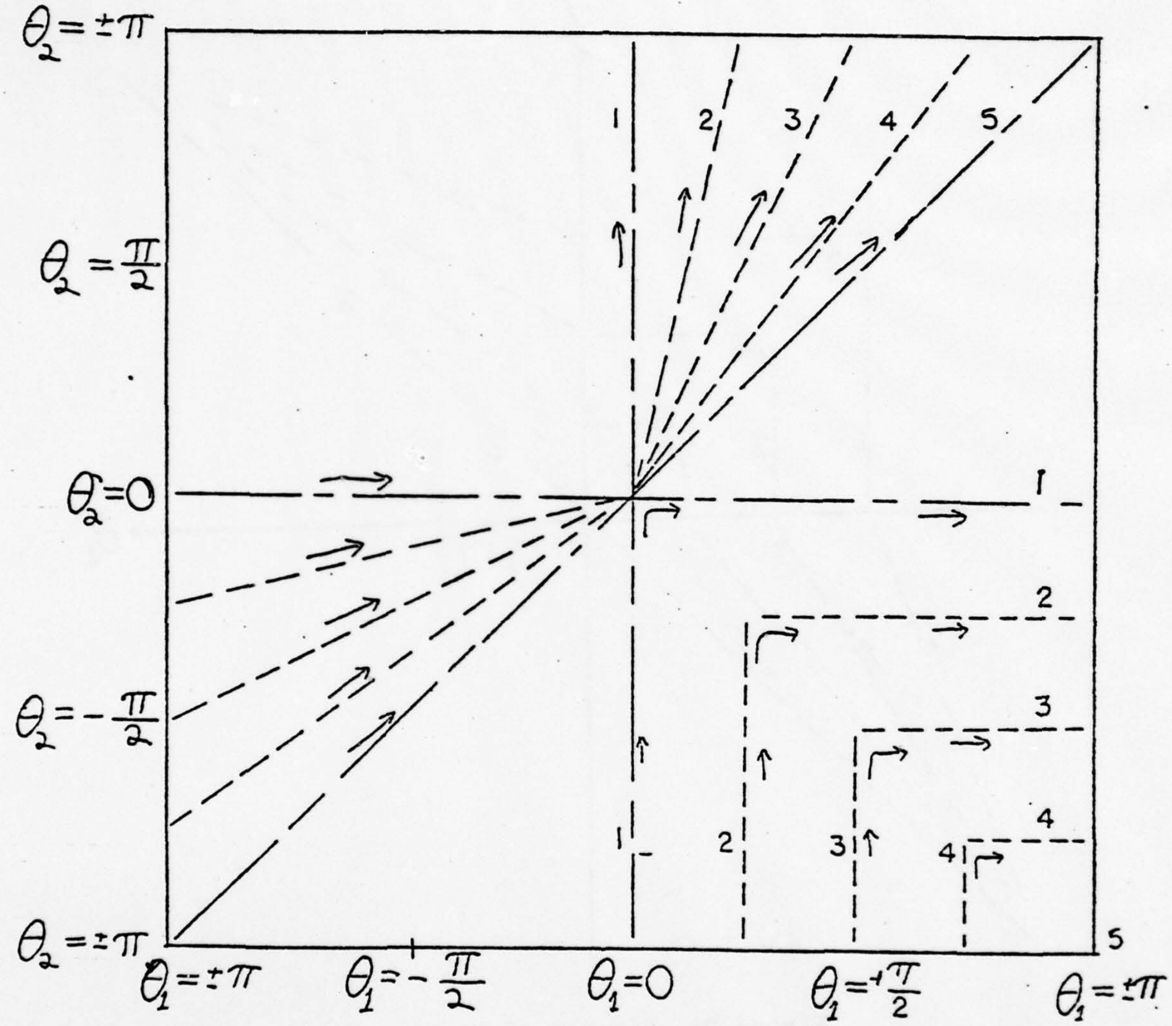

Figure 2: Diagram of homotopy.

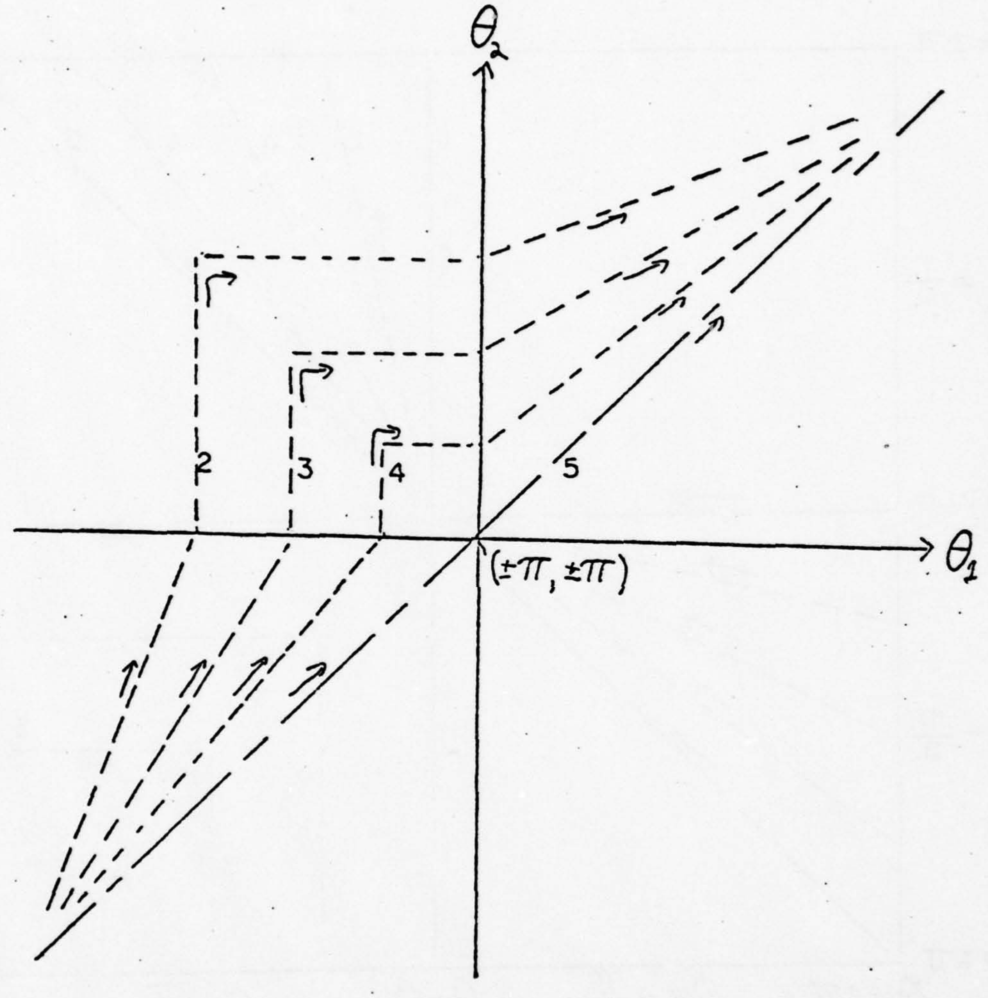

Figure 3: Diagram of homotopy.

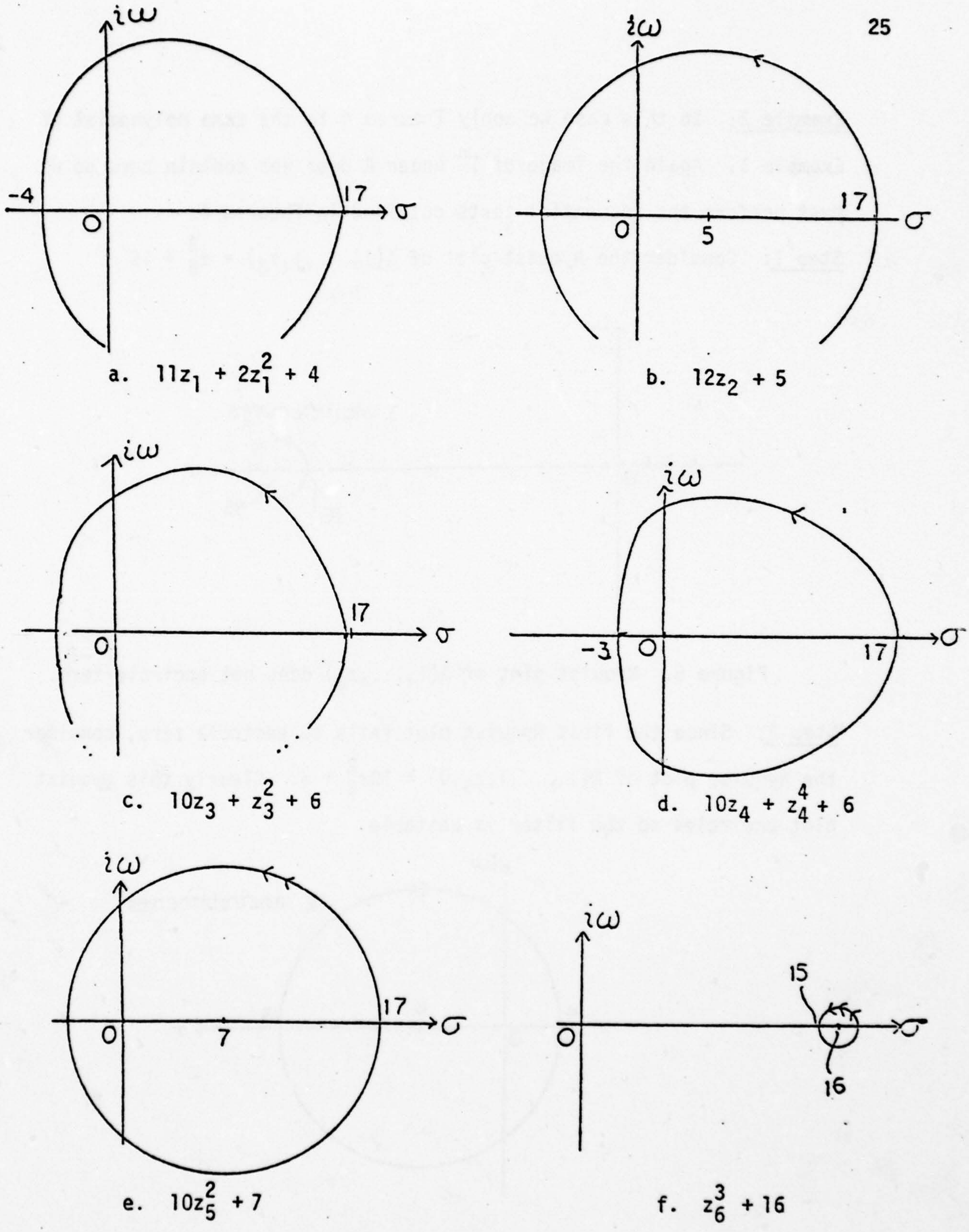

Figure 5. Nyquist plots for the six single variable functions of equations 13 through 16.

Example 2: In this case we apply Theorem 4 to the same polynomial of Example 1. Again the image of T^n under A does not contain zero so we must perform the sequential tests outlined in Theorem 4.

Step 1: Consider the Nyquist plot of $A(1,...,1,z_6) = z_6^3 + 16$

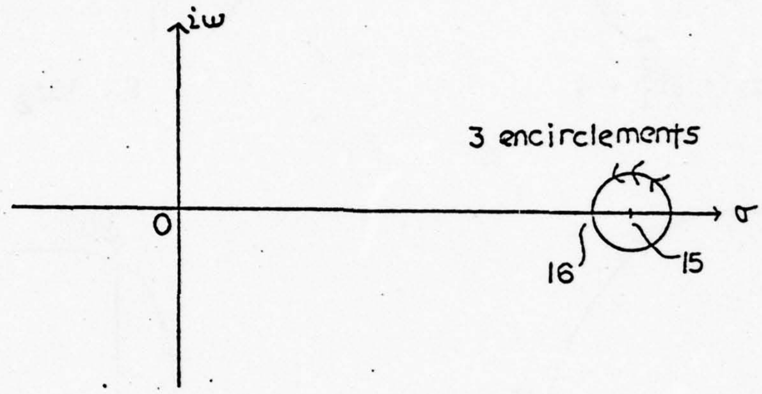

Figure 6. Nyquist plot of $A(1,...,z_6)$ does not encircle zero.

Step 2: Since the first Nyquist plot fails to encircle zero, consider the Nyquist plot of $A(1,...,1,z_5,0) = 10z_5^2 + 6$. Clearly this Nyquist plot encircles so the filter is unstable.

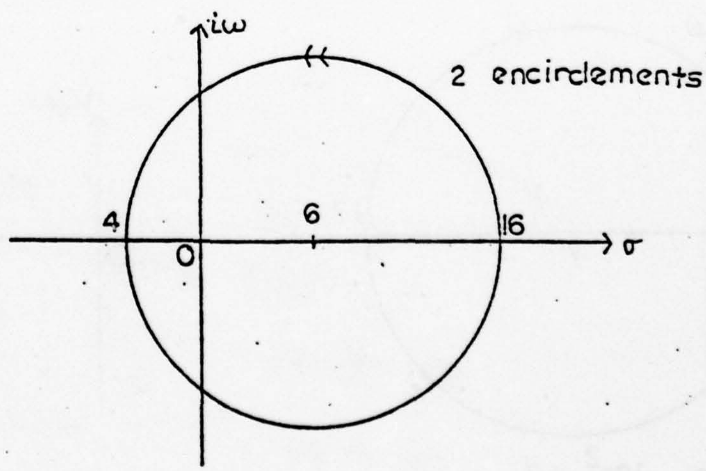

Figure 7. Nyquist plot of $A(1,...,1,z_5,0)$.

<u>Example 3</u>: In this we apply Theorem 5 to the same polynomial of the previous examples. Notice that in this case the curve is of higher order and appears complicated. There also may be some numerical problems in obtaining an accurate Nyquist plot of this curve.

Clearly $A(T^n) \neq 0$ as in the previous examples. Thus consider the Nyquist plot of $A(z_1,\ldots,z_6) = z^7 + 10z^6 + 3z^3 + 3$ where we have set $z = z_1 = \ldots = z_6$.

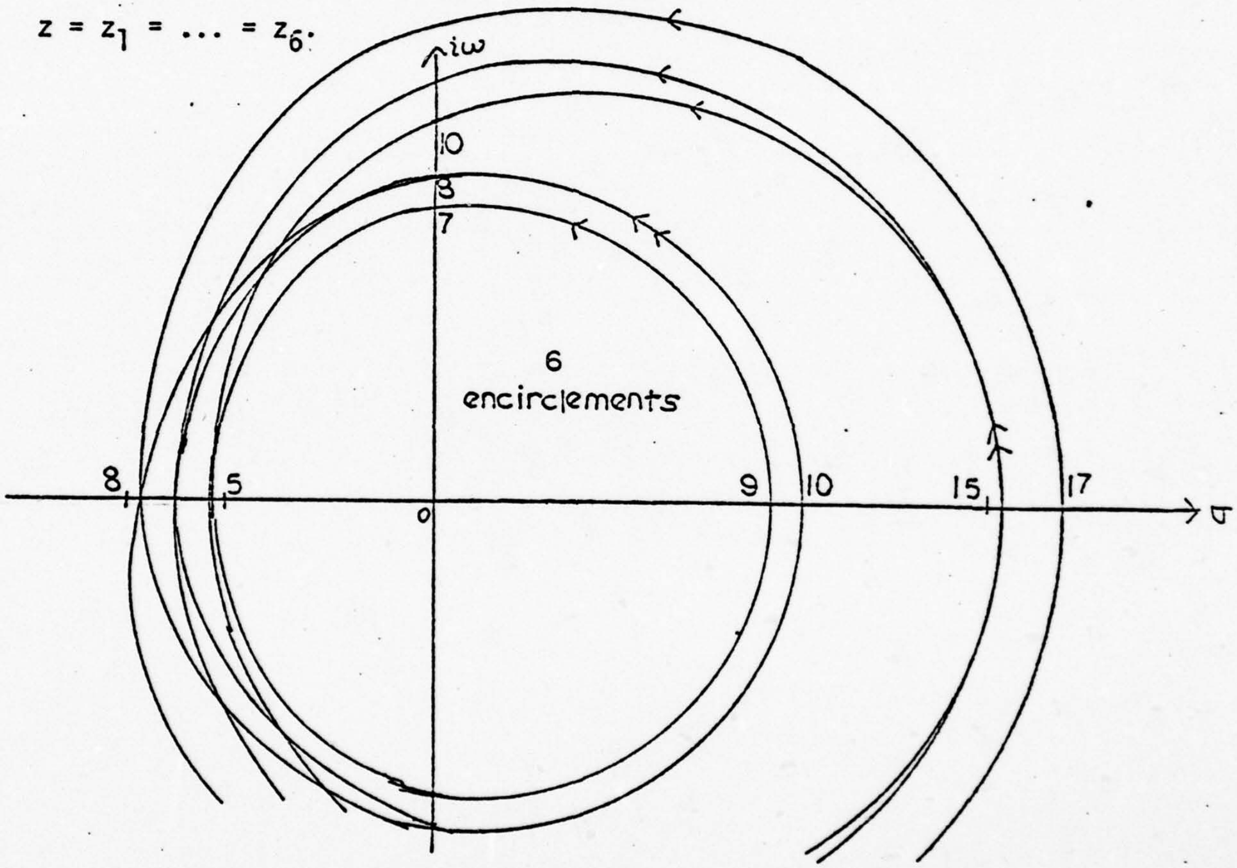

Figure 8: The single Nyquist plot of $A(z,z,z,z,z,z)$

Observe that the number of times the curve of Figure 8 encircles zero equals the number of times all the curves of Figure 5, taken together, encircle zero.

Lastly it is interesting to wonder at the usefulness of these plots for the design engineer. The authors believe that in time these

plots will be shown to supply a large amount of information on the behavior of a system.

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THREE GRAPHICAL TESTS FOR THE STABILITY OF MULTIDIMENSIONAL DIGITAL FILTERS*

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THREE GRAPHICAL TESTS FOR THE STABILITY OF MULTIDIMENSIONAL DIGITAL FILTERS[$]

by

R. DeCarlo, R. Saeks, and J. Murray

Abstract

This paper discusses three graphical tests for determining the stability of multidimensional digital filters characterized by an appropriate transfer function in several complex variables. Each test is carried out as a finite number of "Nyquist" plots in the complex plane.

Introduction

Recently two of the authors constructed an algebraic topological proof of the Nyquist Criterion (2) (3). The value of this rather sophisticated approach has been harvested in generalizations to systems characterized by transfer functions in several complex variables (1) (2) (3), in particular multidimensional digital filters. Specifically the paper illustrates three graphical tests, similar to the classical Nyquist test, carried out in the complex plane, which determine the stability of a multidimensional digital filter with a transfer function $H(z_1,\ldots,z_n) = B(z_1,\ldots,z_n)/A(z_1,\ldots,z_n)$ where z_i are complex variables and A and B are relatively prime polynomials. The purpose of the paper is to consider these three tests as applied to two different examples.

Background and Main Theorems

Basic to the theory is the 2n (real) dimensional polydisc (8) which is the $\mathbb{C}^n$ analog of the unit disc of $\mathbb{C}$. Mathematically the poly-

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disc P^n is

$$P^n = \{ (z_1,\ldots,z_n) \text{ in } \mathbb{C}^n \,\big|\, |z_i| \leq 1, \;\; i = 1,\ldots,n\}$$

There are four separate notions of boundary of the polydisc (1)(8). First is the usual topological boundary

$$B^n = \{ (z_1,\ldots,z_n) \text{ in } \mathbb{C}^n \,\big|\, |z_i| \leq 1,\; i = 1,\ldots,n \text{ and } |z_k| = 1 \text{ for some } k \}$$

Second is the distinguished boundary

$$T^n = \{ (z_1,\ldots,z_n) \text{ in } \mathbb{C}^n \,\big|\, |z_i| = 1, \;\; i = 1,\ldots,n\}$$

T^n serves as the multidimensional analog of the $j\omega$-axis. In particular the frequency response (7) of a digital filter is the evaluation of its transfer function over T^n.

Thirdly we have

$$M^n = \{ (z_1,\ldots,z_n) \text{ in } \mathbb{C}^n \,\big|\, |z_i| = 1,\; i = 1,\ldots,k-1,k+1,\ldots,n; \;\; |z_k| \leq 1 \}$$

where k ranges from 1 through n. This is a boundary notion in the sense that n-1 coordinates take on extremal values. Finally, the last notion of "boundary" is

$$H^n = \{ (z_1,\ldots,z_n) \text{ is } \mathbb{C}^n \,\big|\, |z_i| = 1, \;\; i = 1,\ldots,k-1; \;\; |z_k| \leq 1;\; z_i = 0,\; i = k+1,\ldots,n\}$$

where again k varies from 1 to n. The importance of this concept was first noted by Huang (5). Later it was generalized in (7).

With these notions of boundary one may prove the following Theorem. The proof, however, of the following equivalences is found in the references (1) (2) (5) (7) (9).

Theorem 1: Let a causal multidimensional digital filter be characterized by a rational transfer function in several complex variables. Assume the numerator and denominator polynomials are relatively prime. Then the following are equivalent stability conditions:

(i) the pole set of the transfer function has a null intersection with P^n

(ii) the pole set of the transfer function has a null intersection with B^n

(iii) The pole set of the transfer function has a null intersection with H^n

(iv) the pole set of the transfer function has a null intersection with M^n.

The trouble with these conditions is that the actual test is carried out in a higher dimensional space. For example P^n is 2n dimensional, B^n is (2n-1) dimensional, while H^n and M^n are both (n+1) dimensional. Intuitively, the equivalences of this theorem follow because the pole set of a rational function in several complex variables is an infinite continuum which must intersect the different boundaries of the polydisc if it intersects the polydisc at all.

With the intuition gained in (2) (3) (7) the authors were able to simplify these results to graphical tests in the complex plane. The following three theorems are the fruit of this endeavor. Before stating

these theorems, one final definition is in order.

The "Nyquist plot" of a polynomial, f('), in one complex variable is defined to be the image of T^1 under the map f(') where T^1 is the unit circle of the complex plane. With this in hand, we have the following three tests. Again the proofs can be found in the references (1) (2) (3) (9).

Theorem 2: Let A be the denominator polynomial of a multidimensional digital filter as characterized in Theorem 1. A is a polynomial mapping $\mathbb{C}^n$ to $\mathbb{C}$. Then A has no zeros in P^n (i.e. the filter is stable) if and only if

(i) A has no zeros on T^n, and

(ii) The Nyquist plots for the single variable functions $A(1,\ldots,1,z_k,1,\ldots,1)$ $k = 1,\ldots,n$ do not encircle zero.

Theroem 3: Let A be as above. Then A has no zeros in P^n (i.e. the filter is stable) if and only if

(i) A has no zeros on T^n, and

(ii) The Nyquist plots for the single variable functions $A(1,\ldots,1,z_k,0,\ldots,0)$ $k = 1,\ldots,n$, do not encircle zero.

Theroem 4: Let A be as above. Then A has no zeros in P^n (i.e. the filter is stable) if and only if

(i) A has no zeros on T^n, and

(ii) The Nyquist plot for the single variable function $A(z,z,\ldots,z)$ does not encircle zero

Each of these tests has essentially the same two parts. First one performs the appropriate encirclement test(s); if zero is not encircled, one then proceeds to check the image of the distinguished boundary.

This order of testing (encirclement first, then frequency response) seems in most cases to be preferable to the reverse order, since much less computation is involved in the encirclement tests; however, in cases where the frequency response is known a priori, or must be plotted in any case, the order is immaterial.

It might appear that the third test (Theorem 4) is the best, since it involves only one encirclement test; however, in many cases the relative complexity of the polynomial $A(z,z,\dots,z)$ will more than offset this advantage. Similarly, in many cases, Theorem 3 may be much easier to apply than Theorem 2. (This is illustrated in the first example) Theorems 2 and 4, however, do have two advantages. The first is mainly philosophical; these Theorems give a test for stability purely in terms of the frequency response of the function A, which corresponds closely with the idea of the Nyquist criterion in one variable. The second advantage is that by filling in the interior of the encirclement plot(s) and taking this region together with the image of the distinguished boundary, one obtains the image of the entire polydisc, from which one can get an accurate idea of stability margins. (The point here is that we have found the image of a 2n-dimensional set--the polydisc--by plotting an n-dimensional set and a 1-dimensional set).

EXAMPLES

In this section we apply each of the above tests to two examples.

Example 1:

$$A(z_1,z_2) = 5/4\, z_1^2 z_2^2 + 1/2\, z_1 z_2^3 + 1/2\, z_1^3 z_2 + 3\, z_1 z_2^2 + 3\, z_1^2 z_2$$
$$- z_1^2 - z_2^2 + 3\, z_1 z_2 - 2\, z_1 - 2\, z_z + 1$$

In this case, we have

$$A(z,1) = 1/2\, z^3 + 13/4\, z^2 + 9/2\, z - 2$$

$$A(1,z) = 1/2\, z^3 + 13/4\, z^2 + 9/2\, z - 2.$$

These polynomials are identical; the image of the unit circle being given in Fig. 1(a).*

Since this curve encircles the origin, we deduce immediately that the filter is unstable; for purposes of illustration, we will carry out the other tests.

$$A(z,0) = -z^2 - 2z + 1$$

$A(1,z)$ is as before (Fig. 1(a)); $A(z\ ,0)$ is plotted (for $z\ = e^{i\theta}$) in Fig. 1(b). Again, either plot suffices to verify instability, and clearly $A(z,0)$ gives the simpler test.

To apply the third test, we calculate

$$A(z,z) = 9/4\, z^4 + 6\, z^3 + z^2 - 4z + 1$$

and the image of the unit circle under this mapping is plotted in Fig. 1(c). The relative complexity is apparent; however, it again verifies instability. Finally, we plot the image of the distinguished boundary in Fig. 1(d); it can be seen that it does not include the origin, although it does in some sense "encircle" it.

The second example shows that this last kind of "encirclement" is irrelevant; nothing can be deduced from it.

*(This illustrates the obvious fact that if the polynomial is symmetric in $z_1,\ldots,z_n$, then the n plots in Theorem 2 in fact reduce to 1 plot--usually simpler than the plot in Theorem 4. Such symmetry is quite common).

Example 2:

$$A(z_1,z_2) = (z_1 + 2)^3 (z_2 + 2)^3$$

As before, the plots for A(z,1) and A(1,z) are identical

$$A(z,1) = 27(z + 2)^3$$

$$A(1,z) = 27(z + 2)^3;$$

This plot is given in Fig. 2(a); it does not encircle 0.

In this case, the plot for $A(z,0) = 8(z + 2)^3$ differs from the previous plot only by a scale factor; we do not draw it separately.

Finally, the plot for $A(z,z) = (z + 2)^6$ is given in Fig. 2(b); again it does not encircle the origin.

Thus, in order to determine stability in this case, it is necessary to plot the image of the distinguished boundary; this is done in Fig. 2(c). Since this image does not contain the origin (although it does surround it), we conclude that the filter is stable. This of course is obvious analytically; the present example is merely to illustrate the tests.

Note: Because of the magnitudes of the numbers involved, Figs.2(a) - 2(c) are not drawn to scale.

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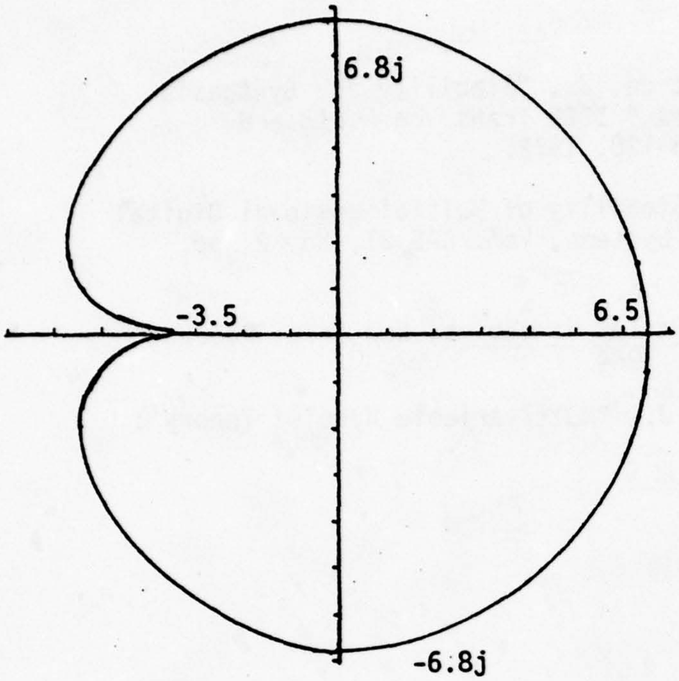

Fig. 1(a)

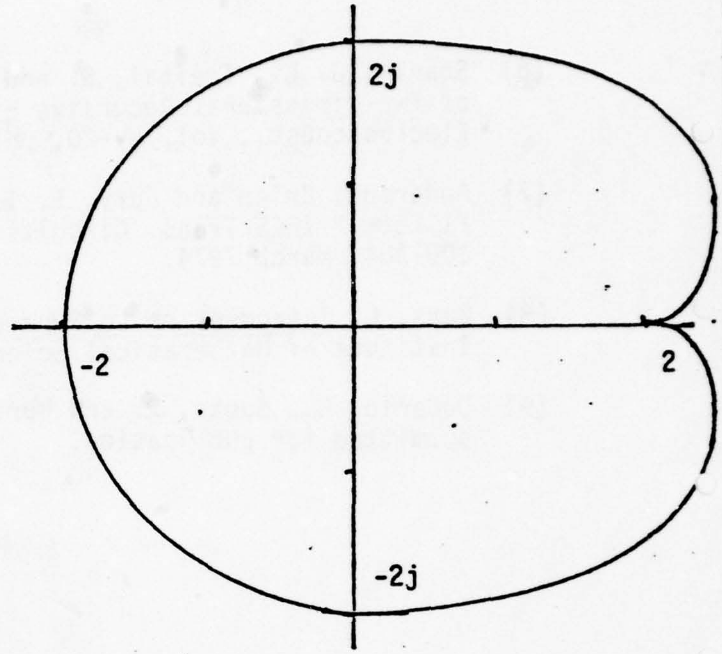

Fig. 1(b)

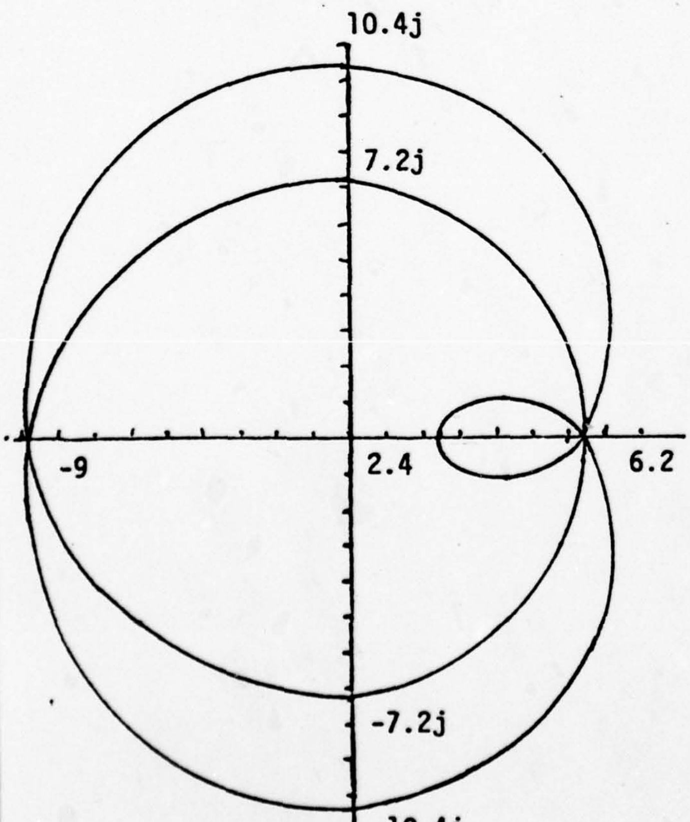

Fig. 1(c)

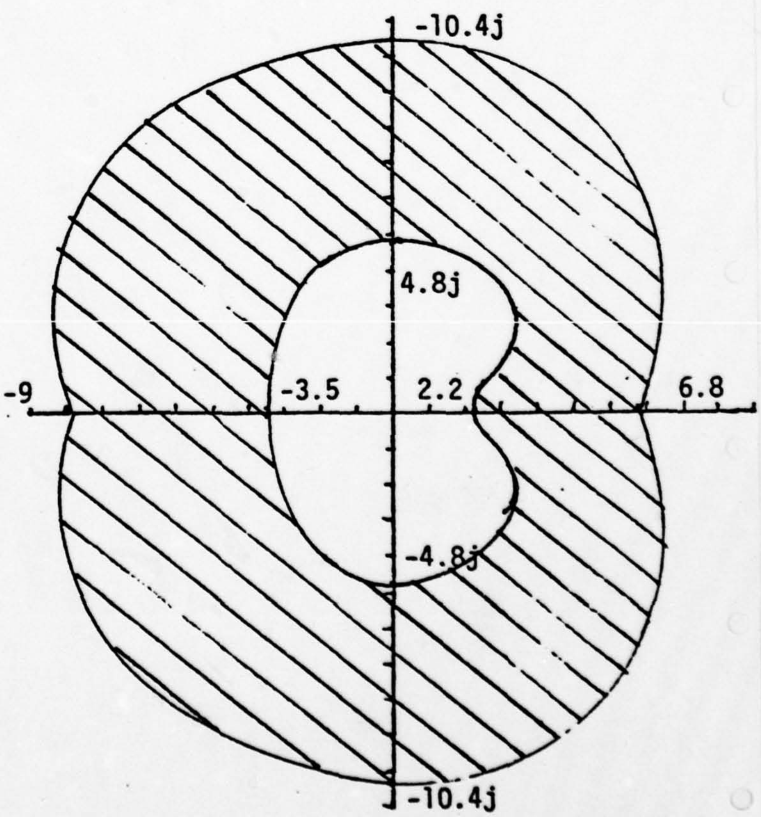

Fig. 1(d)

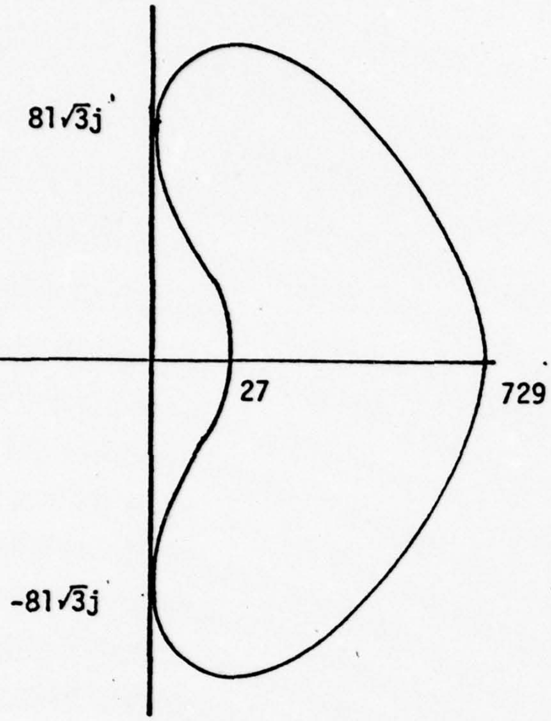

Fig. 2(a)

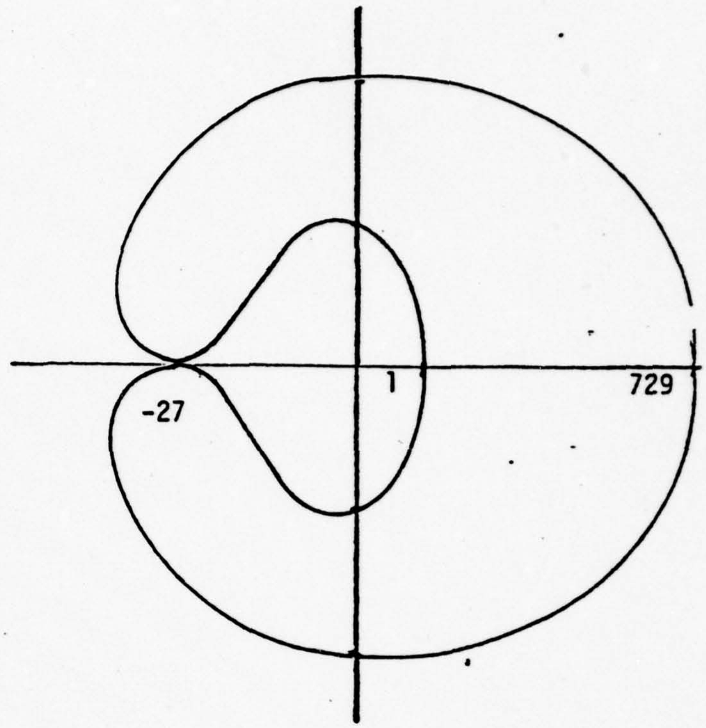

Fig. 2(b)

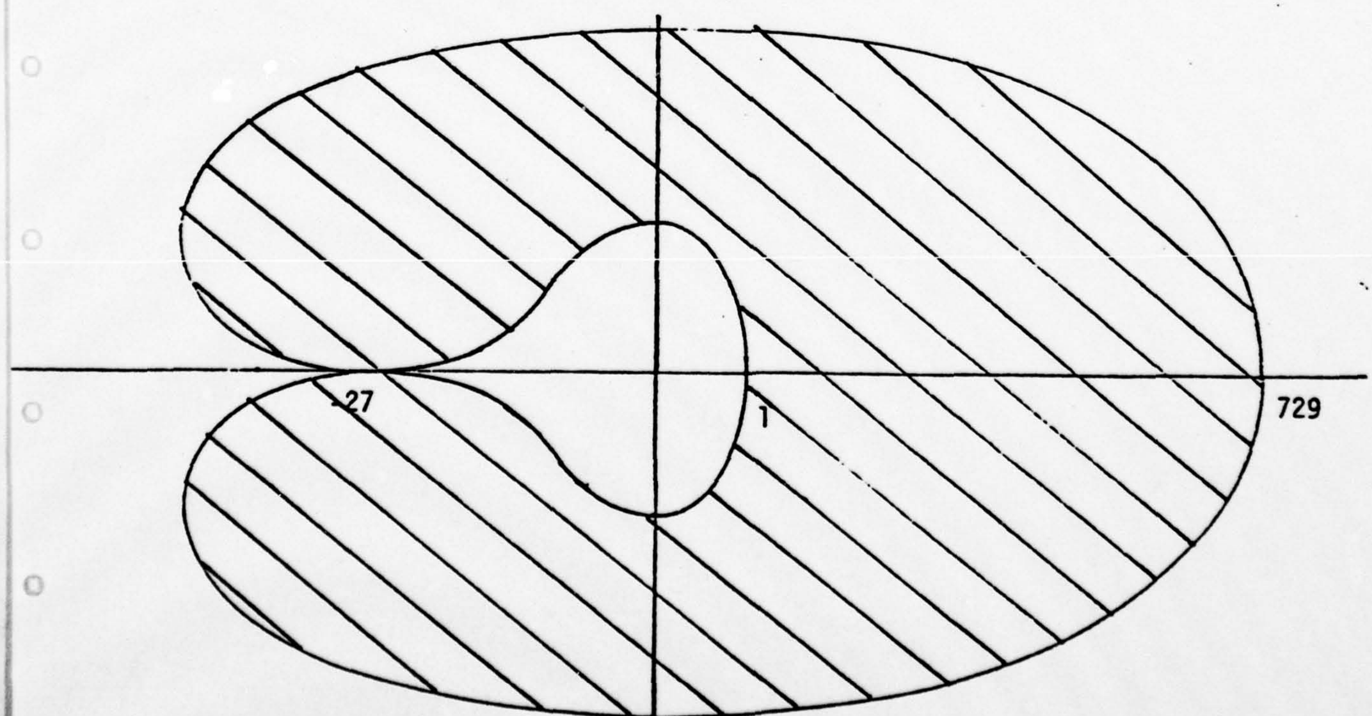

Fig. 2(c)

A NYQUIST-LIKE TEST FOR THE STABILITY OF TWO DIMENSIONAL DIGITAL FILTERS*

R.A. DeCarlo, R. Sakes and J. Murray

*This research supported in part by AFOSR grant 74-2631.

A NYQUIST-LIKE TEST FOR THE STABILITY OF TWO DIMENSIONAL DIGITAL FILTERS*

by

R. DeCarlo, R. Saeks, and J. Murray

ABSTRACT: This paper constructs a Nyquist-like test for the stability of two dimensional digital filters. The test takes the form of a continuum of classical one variable Nyquist plots parameterized by the elements of the unit circle of the complex plane. Since the parameter space is compact the test can be accurately approximated by a finite number of classical one variable Nyquist plots and is therefore readily implemented on a computer.

A two dimensional digital filter is characterized by a rational transfer function in two complex variables

1. $$\frac{B(z_1,z_2)}{A(z_1,z_2)}$$

where $A(z_1,z_2)$ and $B(z_1,z_2)$ are relatively prime polynomials in z_1 and z_2. For the purpose of this paper we say that the digital filter is stable if $A(z_1,z_2) \neq 0$ for $|z_1| \leq 1$ and $|z_2| \leq 1$. This <u>structural stability</u> condition implies that the filter is bounded-input bounded output stable though as recently shown by Goodman[1] the condition is actually slightly stronger. Huang showed that this 4-dimensional stability condition was actually equivalent to the 3-dimensional condition that $A(z_1,z_2) \neq 0$ for $|z_1| = 1$ and $|z_2| \leq 1$ or $|z_1| \leq 1$ and $z_2 = 0$ which we use as the basis of our theory.

The key to the formulation of our Nyquist-like theory is the observation that from an abstract analytic function point of view the classical one variable Nyquist plot is simply a method of determining whether or not

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an analytic function in one variable has zeros in an appropriate region by plotting the image of the function on the boundary of the region. To obtain a Nyquist theory in two variables we therefore decompose the region of C^2 in which $A(z_1,z_2)$ is forbidden to have zeros by Huang's theorem into the union of a family of one variable regions to which the classical Nyquist theorem applies. More precisely, for each complex number α of magnitude one we define the disk D_α in C^2 by

2. $$D_\alpha = \{(e^{i\alpha}, z_2) \; ; \; |z_2| \leq 1\}$$

and we define the disk D_0 by

3. $$D_0 = \{(z_1, 0) \; ; \; |z_1| \leq 1\}$$

Now, Huang's theorem may be restated as "the digital filter is stable if and only if $A(z_1,z_2)$ has no zeros in the disks D_0 and D_α $|\alpha| = 1$". Observing that each disk is fixed in one of its variables the polynomial $A(z_1,z_2)$ restricted to any of the above defined disk's is an analytic function of one variable and hence the classical Nyquist test can be used to check for zeros within the disk. In particular, $A(z_1,z_2)$ has zeros in the disk D_α if and only if the Nyquist plot for the one variable function $A(e^{i\alpha},z_2)$ does not equal or encircle zero. Similarly, $A(z_1,z_2)$ has no zeros in the disk D_0 if and only if the Nyquist plot for the one variable function $A(z_1,0)$ does not equal or encircle zero. Combining these observations we obtain the following stability theorem.

<u>Theorem</u>: A digital filter characterized by the two variable transfer function

$$\frac{B(z_1,z_2)}{A(z_1,z_2)}$$

where $A(z_1,z_2)$ and $B(z_1,z_2)$ are relatively prime polynomials in two variables is stable (structurally stable) if and only if the Nyquist plots for the family of one variable functions

$$A(e^{i\alpha},z_2)\ ;\ |\alpha| = 1$$

and

$$A(z_1,0)$$

do not equal or encircle zero.

Although the theorem formally implies that one check a continuum of Nyquist plots parameterized by the complex numbers of magnitude one, in fact, since this set of numbers is compact one can obtain a test with arbitrarily good resolution using only a finite number of plots. Indeed, in a somewhat different context the authors have shown that a similar continuum of Nyquist plots can actually be reduced to a single plot without inducing any error into the stability test.[2] The following examples are based on a finite approximation to the continuum of plots required by the theorem.

Example 1: Let the transfer function of a digital filter be

4. $$H(z_1,z_2) = \frac{1}{1 + .25z_1 + .25z_2} = \frac{B(z_1,z_2)}{A(z_1,z_2)}\ .$$

Step 1: Draw the Nyquist plot for $A(z_1,0)$. This curve, shown in Figure 1, does not encircle zero. So we proceed to the next step as outlined in the theorem.

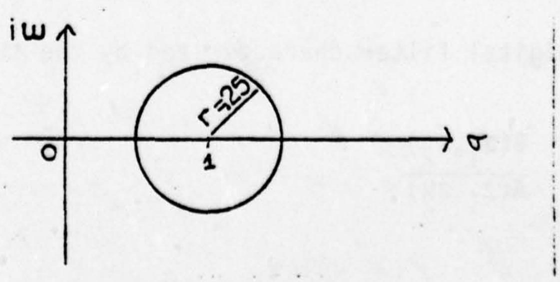

Figure 1: Nyquist plot of $A(z_1,0)$ for example 1.

Step 2: Now consider the family of Nyquist plots for the functions $A(e^{i\alpha}, z_1)$; $|\alpha| = 1$. This family of curves does not encircle "0" as indicated in Figure 2. Thus the filter is stable.

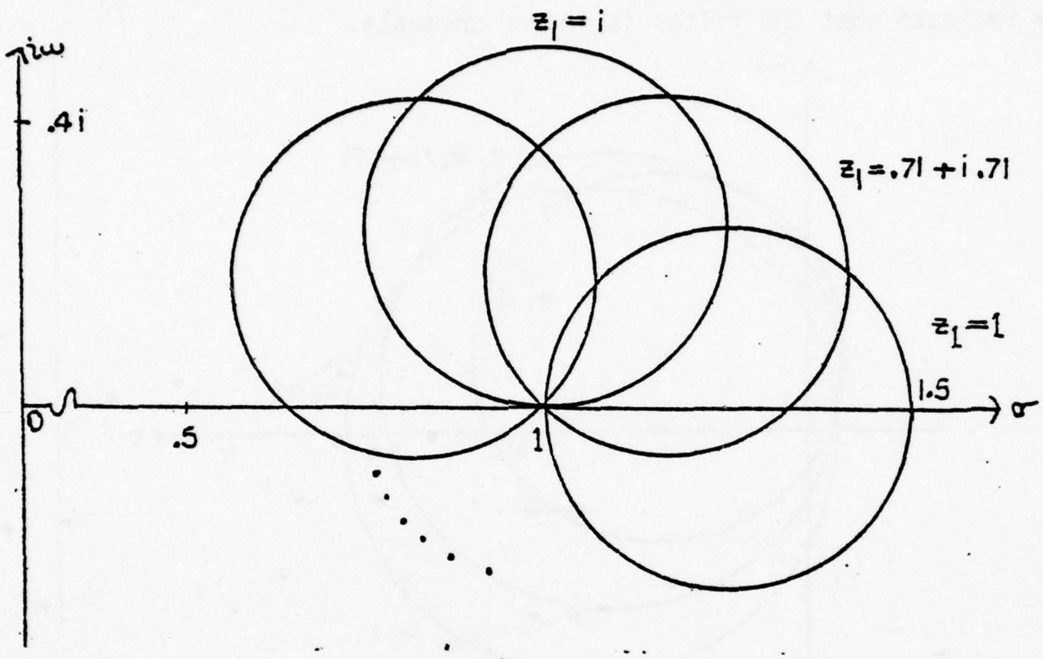

Figure 2: Nyquist plots for $A(e^{i\alpha}, z_2)$ for example 2.

Example 2: Now consider the filter whose transfer function is

$$H(z_1, z_2) = \frac{1}{1 + .5z_1 + .5z_2 + 1.2z_1z_2} = \frac{B(z_1, z_2)}{A(z_1, z_2)} \quad (5.)$$

Step 1: Consider $A(z_1, 0)$. This Nyquist plot is illustrated in Figure 3 and does not encircle zero.

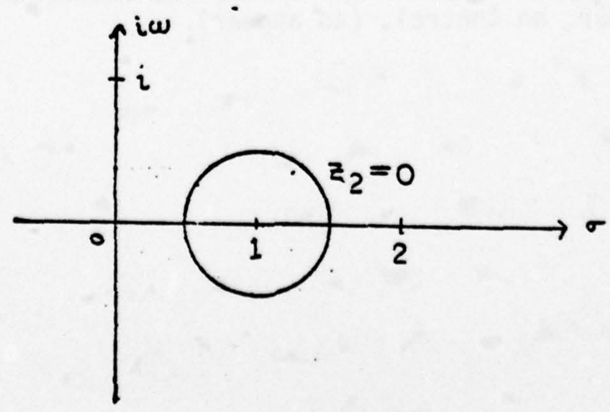

Figure 3: Nyquist plot for $A(z_1, 0)$ for example 2.

At this point no decision can be made so proceed to step 2.

Step 2: Consider the family of functions $A(e^{i\alpha},z_2)$; $|\alpha| = 1$. Nyquist plots for some of these functions are shown in Figure 4. They indicate that the filter is indeed unstable.

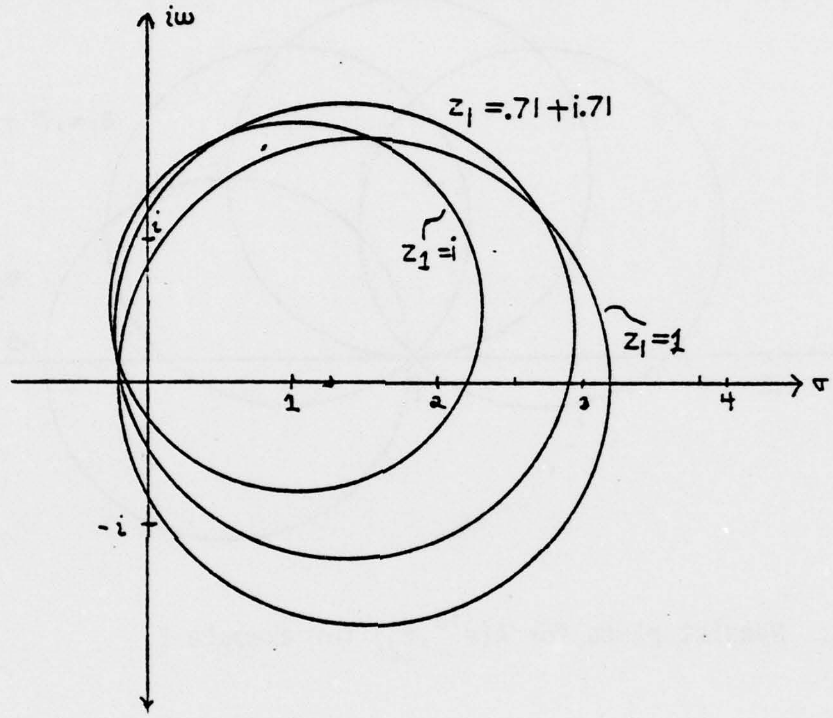

Figure 4: Nyquist plots for $A(e^{i\alpha},z_2)$ for example 2.

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ANOTHER PROOF AND SHARPENING OF HUANG' THEOREM*

J. Murray

*This research supported in part by AFOSR grant 74-2631.

Another Proof and a Sharpening of Huang's Theorem

By

John Murray

ABSTRACT: Two further proofs of Huang's Theorem on the zeros of analytic functions of two variables are given; the first is similar to previous proofs, but is made shorter by the use of a known maximum modulus principle; the second is completely different, using a theorem of Rudin which actually gives a sharper result than Huang's. Finally it is indicated how a correspondingly sharper result may be obtained in higher dimensions.

In [5] Huang stated and gave an incomplete proof of the following theorem:

THEOREM:

A two-variable polynomial $P(Z_1,Z_2)$ has a zero in the polydisc $\overline{U}_2 = \{(Z_1,Z_2) \mid |Z_1| \le 1,\ |Z_2| \le 1\}$ if and only if it has a zero in the set $H_2 = \{(Z_1,Z_2) \mid |Z_1| = 1,\ |Z_2| \le 1\} \cup \{(Z_1,0) \mid |Z_1| \le 1\}$

Two proofs of the Theorem have appeared [2,3], both appealing to the maximum modulus principle; the first proof here is similar but shorter since it uses a convenient known version of the maximum modulus principle.

We will use the following notation:

$$U = \{Z \mid |Z| < 1\}$$

$$\overline{U} = \{Z \mid |Z| \le 1\}$$

$$T = \{Z \mid |Z| = 1\}$$

$$U^n = U \times U \times \ldots \times U \text{ (n times)}$$

$$\overline{U}^n = \overline{U} \times \overline{U} \times \ldots \times \overline{U} \text{ (n times)}$$

$$T^n = T \times T \times \ldots \times T \text{ (n times)}$$

$$H_2' = \{(Z_1,Z_2) \mid |Z_1| = 1,\ |Z_2| \le 1\}$$

$$H_2'' = \{(Z_1,0) \mid |Z_1| \le 1\}$$

Then $H_2 = H_2' \cup H_2''$.

Proof I:

Assume $P(Z_1,Z_2)$ has a zero (Z_1^a, Z_2^a) in $\overline{U}_2$. Let V denote the connected component of the zero-set of $P(Z_1,Z_2)$ which contains (Z_1^a,Z_2^a) and let $W = V \cap \{(Z_1,Z_2) \big| |Z_1| < 1\}$. If $(Z_1^a,Z_2^a) \notin W$, $(Z_1^a,Z_2^a) \in H_2' \subseteq H_2$; we can therefore assume that $(Z_1^a,Z_2^a) \in W \neq \emptyset$. We consider the function $1/Z_2$ restricted to the analytic set W; if this function is not analytic, there is a point in W such that $Z_2 = 0$, i.e., there is a zero in H_2''. If the function is analytic, then by the maximum modulus principle [4,p. 106] $|1/Z_2|$ can not have a strict maximum in W. Thus, since $|Z_2^a| \leq 1$ and $(Z_1^a, Z_2^a) \in W$, there is a point $(Z_1^b,Z_2^b) \in \overline{W} \cap \{(Z_1,Z_2) \big| |Z_1| = 1\}$ such that $|Z_2^b| \leq 1$. By continuity, $P(Z_1^b, Z_2^b) = 0$; thus (Z_1^b,Z_2^b) is a zero of P in H_2'. Q.E.D.

Before proceeding to the second proof, one further definition is needed: If Ψ: $\overline{U} \to \overline{U}$ is continuous, by Ind $\Psi \circ E$ we will mean the usual winding number of the closed curve $\Psi(e^{2\pi it})$, $0 \leq t \leq 1$, about the origin. (In what follows, this curve will never go through the origin).

Proof II:

We will use the following theorem of Rudin [6,p. 87]

If $\Phi = (\Phi_1,\Phi_2,\ldots,\Phi_n)$: $\overline{U} \to \overline{U}^n$ is a continuous mapping such that $\Phi(T) \subseteq T^n$ and Ind $\Phi_i \circ E > 0$ for all i, then $f(Z_1,Z_2,\ldots Z_n)$ has a zero in $\overline{U}^n$ if and only if $f(Z_1,\ldots,Z_n)$ has a zero in the set $T^n \cup \Phi(\overline{U})$. (Here f is any function analytic on U^n and continuous on $\overline{U}^n$.)

To apply this theorem, we take n = 2, and define $\Phi(Z)$ as follows:

$$\Phi(Z) = \begin{cases} (2Z,0) & , |Z| \leq \tfrac{1}{2} \\ (Z/|Z|, (2|Z|-1)Z) & , 1/2 \leq |Z| \leq 1. \end{cases}$$

Φ is clearly continuous, and

$$\Phi_i(e^{2\pi it}) = e^{2\pi it}, \quad i = 1,2,$$

so $\Phi(T) \subseteq T^2$, and Ind Φ_i oE = 1, i = 1,2.

Thus if $P(Z_1,Z_2)$ has a zero in $\overline{U}^2$, it has a zero in $T^2 \cup \Phi(\overline{U})$; this set is clearly a subset of H_2, and the result follows. Q.E.D.

We note the following: Firstly, we have proved only one direction of the implication: the other is trivial. Secondly, both of the above proofs apply to functions analytic on U^2 and continuous on $\overline{U}^2$ and not merely to polynomials. Thirdly, proof II yields a considerably sharper result than Proof I (or any previous proofs of Huang's Theorem) in that the set which must be tested for zeros is a proper subset of H; in fact dim $(T^2 \cup \Phi(\overline{U})) = 2$, while dim H = 3. Fourthly, a correspondingly sharper version of the higher-dimensional extension of Huang's Theorem [1] can be obtained by defining

$$\Phi(Z): \overline{U} \to \overline{U}^n \quad \text{by}$$

$$\Phi(Z) = \Big(\frac{Z}{|Z|},\ldots,\frac{Z}{|Z|}, \underset{\substack{\uparrow \\ k^{th}\text{ entry}}}{\frac{n}{k}(n|Z| - k + 1)Z}, 0,\ldots,0\Big), \quad \frac{k-1}{n} \le |Z| \le \frac{k}{n}$$

for $1 \le k \le n$.

Finally, other choices of Φ will yield other, possibly simpler, tests; this is explored in [7].

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SPECTRAL FACTORIZATION OF QUARTER PLANE DIGITAL FILTERS*

J. Murray

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ABSTRACT

Two sets of necessary and sufficient conditions for the existence of a polynomial spectral factorization of a given polynomial are proved. These conditions are shown to be necessary and sufficient for the existence of a rational spectal factorization of the spectral function of any rational function and necessary but not sufficient for the existence of a spectral factorization of the rational function itself. Here the term "spectral factorization" is to be understood as a factorization into the product of a function without poles and zeros in the open unit polydisc, and a function without poles or zeros in a region inverse to the open unit polydisc. These conditions are seen to give extremely severe restrictions on the spectral function of the given polynomial or rational function, and hence, on the amplitude response of any possible quarter-plane purely recursive (stable) digital filter. The implications of these restrictions for the design of minimum-phase finite-impulse-response and stable infinite-impulse-response filters are discussed. In particular, it is shown that the difficulties which many researchers have encountered in stabilizing two-dimensional infinite-impulse-response filters are inherent in the problem and can not be avoided by a refinement of technique; any method which attempts to stabilize a filter by finding a stable denominator polynomial whose spectral function matches (or closely approximates) that of a given unstable denominator polynomial must fail for a large class of polynomials. This is because there is no stable polynomial having the same spectral function as a polynomial in this class (nor even approximating it well); thus, it is precisely the attempt to match the amplitude response of the unstable filter which forces the new filter to be unstable also. This is in sharp contrast to the situation in one dimension, where any rational spectral function has a rational spectral factorization.

Introduction:

The subject of two-dimensional digital filters has received considerable attention of late: in particular, two-dimensional spectral factorization has been treated in a number of papers - it is considered in great detail in reference [1]. The major problem which arises is that in general the spectral factors of a rational transfer function are not rational: some further processing, such as truncation and smoothing, is usually employed to yield approximate rational factors. It is, therefore, somewhat surprising that the class of rational functions for which a rational spectral factorization exists does not seem to have been investigated. In this paper, we give two sets of conditions which must be satisfied by such functions (theorems 1 and 3); a converse is given which may be applied to the numerator and denominator polynomials separately. Now, the polynomial spectral factors (when they exist) of a given polynomial are minimum- and maximum-phase polynomials; conversely, every such polynomial gives rise to trivial spectral factors. Motivated by this, we apply the results of theorems 1 and 3 to the particular case of minimum-phase polynomials (i.e., polynomials without zeros in the unit polydisc).

In this context, the main consequences of the results of this paper may be broadly outlined as follows:

i) A given polynomial has exactly the same amplitude response as a minimum-phase polynomial if and only if the classical one-variable method (of factoring the original polynomial into a product of two polynomials devoid of zeros in certain regions) can be applied. (This result is in fact implicit in [1], but does not appear to have been explicitly stated in the literature). The corresponding statement for minimum-phase, stable rational functions is false, however.

ii) If the conditions given in theorems 1 and/or 3 are not satisfied, then not alone is there no minimum-phase stable rational function having exactly the same amplitude response as the original; the original amplitude response can not even be approximated arbitrarily well by minimum-phase stable rational functions. This follows from the fact that the conditions in theorems 1 and 3 are conditions on the amplitude response which are preserved under any reasonable kind of convergence.

iii) The conditions in theorem 3 are easily visualized and surprisingly stringent; they require essentially that the gain of the filter, averaged over certain directions in the frequency plane, have no variation in a perpendicular direction. (see the discussion following theorem 3). This gives extremely severe restrictions on the amplitude response of minimum-phase FIR filters, minimum-phase stable IIR filters, and the denominator polynomial of arbitrary stable IIR filters.

iv) It has been pointed out by Bose [9] and Woods [10], and again is implicit in [1], that there exist purely recursive filters whose amlitude responses are not realizable as the amplitude response of any stable purely recursive filter, and that consequently any stabilization method which attempts to match the amplitude response of the original filter is doomed to failure. The restrictions referred to in iii), above, reinforce this conclusion and identify the precise properties of the examples in [9] and [10] which make stabilization impossible.

Definitions and Notation:

Our notation will follow that in [2]; we repeat it here for convenience. For simplicity we restrict ourselves throughout to two dimensions, although

there does not appear to be any difficulty in extending the results to higher dimensions. Thus all functions are assumed throughout to be rational functions of two complex variables unless otherwise stated; we further exclude the zero function. Two-dimensional complex space will be denoted by $\mathbb{C}^2$,i.e, $\mathbb{C}^2 = \{(Z_1, Z_2) | Z_1$ and Z_2 are complex numbers$\}$. The open unit polydisc will be denoted by U^2 ,i.e.,

$$U^2 = \{(Z_1,Z_2)\varepsilon\ \mathbb{C}^2 \mid |Z_1| <1 \text{ and } |Z_2| <1\}$$

and its closure will be denoted by $\overline{U}^2$:

$$\overline{U}^2 = \{(Z_1, Z_2)\varepsilon\ \mathbb{C}^2 \mid |Z_1| \leq 1 \text{ and } |Z_2| \leq 1\}$$

The distinguished boundary of the unit polydisc will be denoted by T^2:

$$T^2 = \{(Z_1, Z_2)\varepsilon\ \mathbb{C}^2 \mid |Z_1| = 1 \text{ and } |Z_2| = 1\}$$

The frequency response of the filter whose transer function is $f(Z_1,Z_2)$ is simply the restriction of f to T^2. We will find it convenient to denote this restriction by f^*.

The one-dimensional sets corresponding to the above are:

$$U = \{Z \ \varepsilon\ \mathbb{C} \mid |Z| <1\}$$

$$\overline{U} = \{Z \ \varepsilon\ \mathbb{C} \mid |Z| \leq 1\}$$

$$T = \{Z \ \varepsilon\ \mathbb{C} \mid |Z| =1\}$$

We need one further subset of $\mathbb{C}^2$:

$$V^2 = \{(Z_1,Z_2) \ \varepsilon\ \mathbb{C}^2 \mid |Z_1| >1 \text{ and } |Z_2| >1\}.$$

By the Fourier coefficients of a function h (θ_1,θ_2) defined on T^2 we mean the numbers

$$a_{mn} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} h(\theta_1,\theta_2)\, e^{-j(m\theta_1 + n\theta_2)} d\theta_1\, d\theta_2 .$$

Finally, let us state precisely what we mean by the term spectral factorization. Several different forms of spectral factorization are treated in [1]; here we will be concerned only with the simplest form: if f is a rational function, it will be said to have a (rational, quarter-plane) spectral factorization if $f = f_1 f_2$ where f_1 and f_2 are rational functions, f_1 has no poles or zeros in U^2, and f_2 has no poles or zeros in V^2. Several comments are in order concerning this definition:

i) By "rational" we mean only "finite-order"; i.e., the functions are assumed to be expressible as the quotient of two (finite-order) polynomials.

ii) *The quarter-plane property enters only in connection with the* regions in which the factors are assumed to be zero- and pole-free; in particular, if f has no poles or indeterminacies on T^2, and has a quarter-plane spectral factorization, then there is a quarter-plane causal, stable filter whose amplitude response is equal to $|f^*|$.

iii) It would possibly be more natural to work with $\overline{U}^2$ and $\overline{V}^2$ rather than U^2 and V^2 (especially when considering stability). However, to do so would complicate the statements of the theorems considerably, and it is usually clear whether or not the results will hold with $\overline{U}^2$ and $\overline{V}^2$ in place of U^2 and V^2. (One needs only to check for zeros and *poles on* T^2). *In general*, if the "closed" version is not obvious, it is not true; $1 - Z_1 Z_2$ will serve as a counterexample in all such cases.

iv) To simplify the statements of the theorems, the definition has been given in terms of the rational function f itself, rather than the spectral function $|f^*|^2$; however, the conditions given in the theorems actually involve only $|f^*|^2$.

v) We note that V^2 is defined to be a subset of $\mathbb{C}^2$; thus the behaviour of functions at infinity is irrelevant to our purposes.

Spectral Factorization:

Our first criterion for the existence of rational spectral factors is very much in the spirit in which spectral factorization is treated in [1]; it is a trivial consequence of theorem 5.4.7 in [2].

Theorem 1:

If a rational function f on $\mathbb{C}^2$ has a rational spectral factorization then the Fourier coefficents a_{mn} of log $|f^*|$ are zero for all pairs of integers (m,n) such that $m \neq o$, $n \neq o$, and m and n have different signs - that is, for all integer points in the second and fourth quadrants The converse is true for polynomial f.

As mentioned above, this criterion involves only the absolute value of f; it follows that the existence of spectral factors imposes restrictions on the amplitude response of a two-dimensional filter - in contrast with the situation in one dimension. The above criterion, however, does not present these restrictions in an easily visualized form - for instance, it is difficult to guage exactly how severe the restrictions are. For this reason, we next present conditions which are stated in terms of the log-amplitude response itself, rather than its Fourier coefficients. This result takes an approach which seems to differ substantially from those previously known; it gives easily visualized necessary conditions on those rational functions which admit a rational spectral factorization. Before we state this theorem, however, we first present a simple result which will

be used in the proof, and is also of separate interest; one of its consequences is that when rational spectral factors exist, the usual one-dimensional stabilization method (for unstable denominator polynomials) can be used.

Theorem 2:

If the rational function f admits a rational spectral factorization, then there is a rational function $\tilde{f}$ (with deg $\tilde{f} \leq$ deg f) such that

$$|\tilde{f}^*| = |f^*|$$

and $\tilde{f}$ has no poles or zeros in U^2.

Again, the converse holds for polynomial f.

Thus, if the denominator polynomial of an unstable filter has polynomial spectral factors, there is a stable filter of at most the same order with the same amplitude response (provided the polynomial has no zeros on T^2).

Again most of the proof is contained in [2]; we fill in the details here: suppose f has rational spectral factors, then $f = f_1 \frac{P}{Q}$ where f_1 has no poles or zeros in U^2 and P and Q are polynomials without zeros in V^2.

Let $\qquad \tilde{P} = Z_1^m Z_2^n \bar{P}(1/Z_1, 1/Z_2),\ Z_2 \neq 0,\ Z_1 \neq 0$

where m is the degree of P in Z_1, n is the degree of P in Z_2, and $\bar{P}$ is the polynomial whose coefficients are the complex conjugates of the coefficients of P. Clearly $\tilde{P}$ is a polynomial of degree less than or equal to the degree of P, and so is also defined for $Z_1=0$ and $Z_2=0$. Now if $\tilde{P}(Z_1,Z_2) = 0$ for $Z_1 \neq 0$ and $Z_2 \neq 0$, then $\bar{P}(1/Z_1, 1/Z_2) = 0$; this implies that either

$$|1/Z_1| \leq 1 \text{ or } |1/Z_2| \leq 1 \text{ (since P has no zeros in } V^2)$$

and so either

$$|Z_1| \geq 1 \text{ or } |Z_2| \geq 1, \text{ i.e., } (Z_1, Z_2) \notin U^2$$

Thus the only possible zeros of $\tilde{P}$ in U^2 are for $Z_1 = 0$ or $Z_2 = 0$. But by standard results in the theory of several complex variables [8], if the zero-set were nonempty, this would imply that either Z_1 or Z_2 was a factor of $\tilde{P}$, which is impossible by our choice of m and n. Thus $\tilde{P}$ has no zeros in U^2. Finally, on T^2

$$|\tilde{P}(Z_1,Z_2)| = |Z_1^m\, Z_2^m\, \overline{P}\,(1/Z_1,1/Z_2)| = |\overline{P}(\overline{Z}_1,\overline{Z}_2)| = |P(Z_1,Z_2)|.$$

$\tilde{Q}$ is defined similarly and has similar properties. Then

$$\tilde{f} = f_1 \frac{\tilde{P}}{\tilde{Q}}$$

clearly has the required properties.

Conversely, suppose f is any polynomial for which there is a rational function $\tilde{f}$ without poles or zeros in U^2 such that

$$|\tilde{f}^*| = |f^*|$$

then $f/\tilde{f}$ is rational and analytic in U , and

$$|(f/\tilde{f})^*| = 1$$

Thus by theorems 5.2.5 and 5.2.6 in [2], $f/\tilde{f} = P/Q$ where P and Q are polynomials, P has no zeros in V^2, and Q has no zeros in U^2. Then

$$f = P\tilde{f}/Q$$

gives a rational (in fact, polynomial) spectral factorization of f.

<u>The Second Criterion:</u>

Our second set of conditions for the existence of a rational spectral factorization is given in the following:

Theorem 3:

If a rational function f on $\mathbb{C}^2$ admits a rational spectral factorization, then

$$\frac{1}{2\pi}\int_0^{2\pi} \log |f(e^{jm\theta}, e^{j(n\theta + \Psi)})| d\theta$$

is a constant independent of Ψ, $(0 \leq \Psi < 2\pi)$, for all integers $m > o$ and $n > o$.

Again, these conditions depend only on the amplitude response of f. The simplest condition is that for m = 1 and n = 1; it can be easily visualized by drawing two adjacent squares in the $\theta_1 \theta_2$ - plane on which the amplitude response is defined (the frequency response extends to the entire $\theta_1\theta_2$ - plane by periodicity), and drawing lines L_i with slope 1 and length $2\pi\sqrt{2}$ on these squares; see figure 1.

Then the condition for m = 1, n = 1 can be restated as: the "average" amplitude of the function f along the line L_i is a constant - that is, it is as independent of the particular line L_i chosen. ("Average" here is to be understood as the geometric mean of the amplitude, or the arithmetic mean of the log-amplitude). Alternatively, we may say that the average level of the amplitude over any line of slope 1 and of length $2\pi\sqrt{2}$ is independent of the position of the line in the $\theta_1\theta_2$-plane. (For example, we could vary the L_i over the dotted square in the direction $\underline{\hat{n}}$). The conditions for higher m and n have a similar interpretation, with a slope of n/m instead of 1, and length $2\pi\sqrt{m^2 + n^2}$ instead of $2\pi\sqrt{2}$; clearly, if m and n are not relatively prime, the corresponding condition is superfluous.

This theorem then gives a striking limitation on the amplitude response of a rational function which admits a rational spectral factorization; even the simplest of the conditions (that for n=m=1) implies that

such a function can not accurately approximate an amplitude which has large variations in overall level in the direction $\hat{\underline{n}}$ shown in figure 1.

Proof of Theorem 3:

In view of theorem 2, it suffices to prove this under the assumption that f has no poles or zeros in U^2. This assumption implies that f has a holomorphic logarithm in U^2. Then, for any integers $m > o$, $n > o$ and any real number Ψ,

$$\log f(Z^m, Z^n e^{j\Psi})$$

is a holomorphic function of one complex variable for $Z \varepsilon U$. Thus

$$\mathrm{Re}(\log f(Z^m, Z^n e^{j\Psi}))$$

is a harmonic function in U, and so by the mean-value property of harmonic functions

$$\frac{1}{2\pi}\int_T \mathrm{Re}\,(\log f(Z^m, Z^n e^{j\Psi}))\, d\theta = \mathrm{Re}\,(\log f(o^m, o^n e^{j\Psi}))$$

$$\text{i.e.,}\ \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\,(\log f(e^{jm\theta}, e^{j(n\theta+\Psi)}))\, d\theta = \mathrm{Re}(\log f(o,o))$$

But $\mathrm{Re} \log w = \log |w|$ for $w \neq o$, and so

$$\frac{1}{2\pi}\int_0^{2\pi} \log |f(e^{jm\theta}, e^{j(n\theta + \Psi)})|\, d\theta = \log |f(o,o)|$$

and the right-hand side is independent of Ψ (and, incidentally, of m and n also).

An obvious question which arises is the extent to which the converses of these results hold. In fact, the converse of Theorem 3 holds for polynomials, and modified converses of both Theorems 1 and 3 hold even for rational functions. The modification takes the *following form*: If the Fourier coefficients of $\log |f^*|$ (where f is a rational function) vanish

for mn < o, then there is a rational function $\tilde{f}$ with rational spectral factors, (equivalently, a rational function without poles or zeros in U^2), such that $|\tilde{f}^*| = |f^*|$. (A similar statement holds for Theorem 3). However, the proofs of these converses involve some technical analytic details, and so are relegated to an appendix.

The modification in the above converses lies, of course, in the fact that we cannot conclude that f itself has rational spectral factors; thus there are some rational functions which can be stabilized without changing the amplitude response but to which the classical 1-variable factorization technique cannot be applied. A simple example of this is the function

$$f(Z_1, Z_2) = \frac{Z_1 + Z_2 - 1}{Z_1 + Z_2 - Z_1 Z_2}$$

Here, $|f^*|$ is identically 1, and so has trivial spectral factors; but f itself clearly does not.

Although the converses of theorems 1 and 3 are proved in the appendix, there is another result related to the converse of Theorem 3; by strengthening the conditon for m=n=1 alone, we can get a stronger converse for polynomials. Before we state this converse, however, we first give a stability criterion (used in the proof of the converse) which, although previously known, [3], has not appeared in the engineering literature. Although not as sharp (in terms of dimension) as some other known criteria [4], it has two advantages which make it useful for theoretical purposes: first, it is given in terms of a one-parameter family of discs without the lower-dimensional test in [5]; and second, unlike most other stability tests, which conclude the nonvanishing of a polynomial on $\overline{U}^2$ from its nonvanishing on some subset of $\overline{U}^2$ which contains T^2, this test allows the polynomial to vanish at some points in T^2, but concludes only that the polynomial does

not vanish on U^2. The criterion is:

Theorem 4:

Suppose a polynomial f has no zeros in the set

$$\{(Z_1, Z_2) \in U^2 \,\big|\, |Z_1| = |Z_2|\};$$

then f has no zeros in U^2.

This is proved in a much more advanced context in [3]; however, it can also be easily proved by applying one of the criteria in [4] to the polydiscs

$$\overline{U}_r^2 = \{(Z_1,Z_2) \in ¢^2 \,\big|\, |Z_1| \leq r, |Z_2| \leq r\}$$

for $o<r<1$.

For the hypotheses imply that f has no zeros on the distinguished boundary of $\overline{U}_r^2$ (for $o<r<1$), and none on the set

$$\{(Z_1,Z_2) \in ¢^2 | Z_1=Z_2\} \cap \overline{U}_r^2$$

Thus by theorem 5 in [4], f has no zeros in $\overline{U}_r^2$ for any $r < 1$, and so f has no zeros in U^2.

We can now state and prove the partial converse to theorem 3.

Theorem 5:

If f is a polynomial with the property that

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left| f(e^{j\theta}, e^{j(\theta+\Psi)}) \right| d\theta = \log |f(o,o)|, \text{ for } o \leq \Psi < 2\pi,$$

then f has no zeros in U^2.

Thus we strengthen the condition for m=1 and n=1 in theorem 3 by specifying that the constant in question is to be log $|f(o,o)|$: it then follows not only that f has rational spectral factors, but that it is actually zero-free in U^2.

By theorem 4, it suffices to prove that f has no zeros in the set

$$\{(Z_1,Z_2) \in U^2 \,\big|\, |Z_1| = |Z_2|\}$$

But this set is the union of the open discs

$$\{(Z_1,Z_2) | Z_2 = e^{j\Psi}Z_1,\ |Z_1|<1\}, \text{for } o \leq \Psi < 2\pi;$$

we therefore wish to prove that f has no zeros in any of these discs; or equivalently that the function f_ψ of one variable defined by $f_\Psi(Z) = f(Z, Ze^{j\Psi})$ has no zeros in the open unit disc. Applying Jensen's formula [6, p.299] for the unit disc to f_ψ, we get

$$\frac{1}{2\pi}\int_0^{2\pi} \log \left|f_\Psi\,(e^{j\theta})\right| d\theta = \log \left|f_\Psi(o)\right| - \sum \log |Z_i|$$

where the summation is over all the zeros (counted with multiplicity) of f_Ψ in the unit disc. Expressing this in terms of f:

$$\frac{1}{2\pi}\int_0^{2\pi} \log \left|f(e^{j\theta}, e^{j(\theta+\Psi)})\right| d\theta = \log \left|f(o,o)\right| - \sum \log |Z_i|$$

and so $\sum \log |Z_i| = 0$.

Since for any Z_i in the open unit disc $\log |Z_i| < o$, the conclusion follows. (It is clear from the proof that we always have

$$\frac{1}{2\pi}\int_0^{2\pi} \log \left|f(e^{j\theta}, e^{j(\theta+\Psi)})\right| d\theta \geq \log \left|f(o,o)\right| \quad ;$$

it follows from this that in fact the apparently weaker condition

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \log \left|f(e^{j\theta_1}, e^{j\theta_2})\right| d\theta_1 d\theta_2 = \log \left|f(o,o)\right|$$

is sufficient to guarantee that f is zero-free in U^2. See [2,p.73]).

<u>Stable IIR Filters and Minimum-phase FIR Filters:</u>

The very close relationship of spectral factorization to the nonvanishing of polynomials in U^2, and thereby to stable IIR filters (via the denominator polynomial) and minimum-phase filters (via the numerator polynomial) is already clear from the previous sections. The force of theorem 2 is that

purely from the point of view of amplitude response, transfer functions having rational spectral factors are equivalent to those without poles or zeros in U^2. Thus the restrictions on amplitude response in theorems 1 and 3 apply to the denominator polynomial of any stable IIR filter; the contribution of the denominator polynomial to the overall amplitude response of the filter (in the case of an all-pole filter, the entire amplitude response) must satisfy the restrictions imposed by theorems 1 and 3. We have, therefore, identified the properties of the amplitude response which make it impossible to stabilize a filter; if the original amplitude response has large overall variation in the "wrong" directions, attempting to find a stable filter which closely matches this response is futile. Close matching of the amplitude <u>forces</u> instability. This has already been shown by example by Bose [9] and Woods [10]: we now see that it is the variations in the amplitude response in the "wrong" directions in their examples which accounts for their behaviour.

It is also of interest to note that, in the Shanks procedure of minimizing

$$\iint |fg-1|^2 \, d\theta_1 d\theta_2$$

over all polynomials f of given degree (where g is the original polynomial), if the allowable f's were restricted to those which have polynomial spectral factorizations, the procedure would yield a polynomial devoid of zeros in U^2. It does not appear that this observation can be used as the basis for a workable stabilization method, however, since the condition that f have polynomial spectral factors is intractably nonlinear in the coefficients of f; and further, in many cases this procedure would yield an f which was only marginally stable. For the same reasons, restricting oneself throughout the design procedure to polynomials which satisfy the condition in theorem 5 does not appear to be a feasible method of ensuring stability.

Examples and Comments:

An example of the behaviour of those polynomials not possessing polynomial spectral factors has already appeared in the literature, although in a different context; we repeat this example here.

$$A(Z_1,Z_2) = 1 -.75Z_1 + .9Z_1^2 + 1.5Z_2 - 1.2Z_1Z_2 + 1.3Z_1^2Z_2 + 1.2Z_2^2 + .9Z_1Z_2^2 + .5Z_1^2Z_2^2$$

This polynomial was studied in [7]; the associated Shanks polynomial was found to be stable but to have a substantially different amplitude response from that of A (for more details, see [7]). The fact that A does not have polynomial spectral factors was established by checking the condition in theorem 3 for m=n=1 and $\Psi = o$, $\Psi = \pi$, with the following results: (correct to nine decimals)

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|A(e^{j\theta}, e^{j\theta})\right|d\theta = .696570700$$

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|A(e^{j\theta}, e^{j(\theta+\pi)})\right|d\theta = 1.134686936$$

As an example of a polynomial with rational spectral factors, we have

$$B(Z_1,Z_2) = 1 + 2.25Z_1 + 2.25Z_2 + .5Z_1^2 + .5Z_2^2 - 6.5Z_1Z_2 - Z_1^2Z_2 - Z_1Z_2^2 - 4Z_1^2Z_2^2$$

This factors into $(1 + .25Z_1 + .25Z_2 + .5Z_1Z_2)(1 + 2Z_1 + 2Z_2 - 8Z_1Z_2)$ the first factor factor having no zeros in U^2, the second none in V^2; reversing the second factor gives a polynomial without zeros in U^2:

$$\tilde{B}(Z_1,Z_2) = (1 + .25Z_1 + .25Z_2 + .5\,Z_1Z_2)(-8 + 2Z_2 + 2Z_1 + Z_1Z_2)$$

$$= -8-2Z_1Z_2 + .5\,Z_1^2 + .5Z_2^2 + 1.25Z_1^2Z_2 + 1.25Z_1Z_2^2 + .5Z_1^2Z_2^2,$$

and $\tilde{B}$ has the same amplitude response as B.

In order to gain some idea of the stringency of the conditions in theorem 3, let us consider the case of an ideal band-pass filter. By an ideal band-pass filter we will mean a filter whose amplitude response is equal to 1 on some subset, A, of the square $0 \leq \theta_1 < 2\pi$, $0 \leq \theta_2 < 2\pi$, and equal to $K \ll 1$ on the complement of A (of course this specification continues over the whole plane by periodicity). This of course is not the amplitude response of any rational function, but in practice for certain shapes of the set A, one may wish to approximate such a response by a rational function. One easily sees that up to a scale factor, the averages in theorem 3 are in this case merely the fraction:

$$\frac{\text{length of the line } L_i \text{ lying in the complement of A}}{\text{Total length of the line } L_i}$$

It is easily seen from this that there are very few passband shapes of practical interest which satisfy even the first of these conditions (where n=1 and m=1); in other words, very few which can be accurately approximated by transfer functions having rational spectral factors. (This is not to imply that one would in practice be restricted to such filters; the above discussion is meant solely as an indication of the severity of the restrictions on the amplitude of such filters).

Finally, we remark that there does not seem to be any difficulty in extending the results in this paper to higher dimensions, and to multidimensional systems other than digital filters.

APPENDIX

The converses to Theorems 1 and 3.

These converses involve some technical ideas and results from [2]; the most important ideas are those of inner function [2,p.105], outer function [2,p. 72], Poisson integral [2, p.17] and the classes $N(U^2)$ [2, p.44] and $N_*(U^2)$ [2, p.44].

We will also use the following notation from [2] (Here f is an analytic function on U):

i. $f^*(e^{j\theta_1}, e^{j\theta_2}) \triangleq \lim_{r \to 1^-} f(re^{j\theta_1}, re^{j\theta_2})$

will denote the radial limit of f

(this is clearly consistent with our previous use of f*);

ii. For $w = (w_1, w_2) \in T^2$, $f_w(Z)$ will denote the one-varible function defined by

$$f_w(Z) \triangleq f(Zw_1, Zw_2);$$

iii. if ϕ is a function defined on T^2 which is absolutely integrable there,

$$\hat{\phi}(m,n) \triangleq \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \exp(-jm\theta_1 - jn\theta_2)\, \phi(\theta_1,\theta_2)\, d\theta_2 d\theta_1$$

will denote the Fourier coefficients of ϕ.

iv. For any function ϕ on T^2,

$$\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \phi(\theta_1,\theta_2)\, d\theta_2 d\theta_1 \text{ will be denoted by}$$

$$\int_{T^2} \phi\, dm \quad \text{or} \quad \int_{T^2} \phi(w)\, d\, m\, (w).$$

We will first prove the converse to Theorem 1, and from this derive the converse to Theorem 3. First of all, however, we need the following

lemma (which is given as a problem in [2]).

Lemma A1:

If ϕ is a real-valued function defined on T^2 such that

$$\phi \in L^1(T^2) \quad (\text{i.e.,} \int_{T^2} |\phi| dm < \infty)$$

and

$$\hat{\phi}(m,n) = 0 \quad \text{for } mn < 0,$$

then there is an outer function f on U^2 such that

$$P[\phi] = \log|f|$$

(where P[ ] denotes "Poisson integral of").

Proof

Let $$a_{mn} = \begin{cases} \hat{\phi}(m,n) & (m,n) \neq (o,o) \\ 1/2\, \hat{\phi}(m,n) & (m,n) = (o,o) \end{cases}$$

and let

$$g(Z_1,Z_2) = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} a_{mn} Z_1^m Z_2^n.$$

This series clearly converges uniformly on compact subsets of U^2, and so defines an analytic function there.

If we let $f = e^g$

then f is analytic in U^2, and

$$\log|f| = \sum_{m=o}^{\infty}\sum_{n=o}^{\infty} a_{mn}\, r_1^m r_2^n \exp(jm\theta_1 + jn\theta_2)$$

$$+ \sum_{m=o}^{\infty}\sum_{n=o}^{\infty} \overline{a_{mn}}\, r_1^m r_2^n \exp(-jm\theta_1 - jn\theta_2)$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \hat{\phi}(m,n) r_1^{|m|} r_2^{|n|} \exp(jm\theta_1 + jn\theta_2)$$

$$= P[\phi] \qquad [2,,p.17]$$

Next we prove that f is outer; we have (for $0<r<1$)

$$\int_{T^2} \log^+|f(rw)|\,dm(w) \leq \int_{T^2} |\log|f(r,w)||\,dm(w)$$

$$= \int_{T^2} |P[\phi](rw)|\,dm(w)$$

$$\leq \int_{T^2} |\phi(w)|\,dm(w) \qquad [2,\text{Thm.}2.1.3(c)]$$

$$< \infty$$

and so $f \in N(U^2)$.

Now f^* exists almost everywhere on T^2 [2,Thm.3.3.5] and $\log|f^*| = \phi$ almost everywhere on T^2 [2,Thm.2.2.1]; thus $\log|f| = P[\log|f^*|]$ and so $f \in N_*(U^2)$ [2,Thm.3.3.5], and $\log|f(0)| = \int_{T^2} \log|f^*(w)|\,dm(w)$.

Thus f is outer.

Q.E.D.

We can now prove the converse to Theorem 1:

Theorem A2:

Let $f(Z_1,Z_2)$ be a rational function ($\not\equiv 0$), and let

$$\phi = \log |f^*|$$

If $\hat{\phi}(m,n) = 0$ for $mn<0$, then there is a rational function g without poles or zeros in U^2 such that $|g^*| = |f^*|$.

Proof:

By Lemma A1, there is an outer function g such that

$$\log |g| = P[\log|f|].$$

This implies

$$\log |g^*| = \log |f^*| \quad \text{almost everywhere on } T^2.$$

Therefore, for almost all $w \in T^2$

$$\log |g_w^* (Z)| = \log |f_w^* (Z)| \quad \text{for almost all } Z \in T \quad [2, \text{Lemma } 3.3.2],$$

and g_w is outer for almost all $w \in T^2$ [2,Lemma 4.4.4].

For any such w, let $Z_1, \ldots, Z_n$ denote the poles, and $Z_{n+1}, \ldots Z_m$ the zeros, of $f_w(Z)$ in U, and let

$$\tilde{f}_w(Z) = \prod_{k=1}^{n} \frac{Z-Z_k}{\bar{Z}_k Z-1} \prod_{k=n+1}^{m} \frac{\bar{Z}_k Z-1}{Z-Z_k} f_w(Z)$$

Then $\tilde{f}_w$ has no poles or zeros in U and is rational; hence, $\tilde{f}_w$ is outer. Since g_w is outer, we have $\tilde{f}_w/g_w$ is outer. Also $|\tilde{f}_w^*| = |f_w^*|$, and so $|\tilde{f}_w^*| = |g_w^*|$ for almost all $Z \in T$. Thus $\tilde{f}_w/g_w$ is inner. But a function which is both outer and inner is a constant of modulus 1, and so

$$g_w = e^{j\Psi} \tilde{f}_w \qquad \text{for some real } \Psi.$$

Thus g_w is rational for almost all $w \in T^2$, and so g_w is rational for all $w \in E$, where $E \subseteq T^2$ is a compact set of positive measure (by the inner regularity of the measure). It follows by [2,Thm. 5.2.2] that g is rational (since the vanishing of a polynomial P on a set of positive measure in T^2 would imply

$$\log |P^*| \notin L^1(T^2)$$

and so $P \equiv 0$.)

Thus g is a rational function without poles or zeros in U^2, and

$$|g^*| = |f^*| \quad \text{almost everywhere in } T^2$$

and so, since g and f are both rational,

$$|g^*| = |f^*| \quad \text{on } T^2.$$

Q.E.D.

We next prove the converse to Theorem 3:

<u>Theorem A3:</u>

Let $f(Z_1, Z_2)$ be a rational function ($\not\equiv 0$) and let

$$\phi = \log |f^*| .$$

If $\frac{1}{2\pi} \int_0^{2\pi} \phi\,(m\theta,\ n\theta + \Psi)d\theta$ is a constant independent of Ψ for each pair (m,n) with $m > o$ and $n > o$ then there is a rational function g without poles or zeros in U^2 such that $|g^*| = |f^*|$.

Proof:

Let $m > o$, $n > o$, and let $\ell \neq o$ be an integer.

Then

$$\int_0^{2\pi} e^{j\ell m\Psi} \int_0^{2\pi} \phi(m\theta, n\theta + \Psi)d\theta \, d\Psi = 0$$

$$\Rightarrow \int_0^{2\pi}\int_0^{2\pi} e^{j\ell m\Psi} \phi\,(m\theta, n\theta + \Psi)d\theta d\Psi = 0$$

Making the change of variables defined by

$$\theta = \frac{1}{m}\theta_1$$
$$\Psi = \theta_2 - \frac{n}{m}\theta_1,$$

we get

$$\frac{1}{m}\int_0^{2\pi}\int_{\frac{n}{m}\theta_1}^{\frac{n}{m}\theta_1 + 2\pi} \exp\,(j\ell m\theta_2 - j\ell n\theta_1)\,\phi\,(\theta_1,\theta_2)d\theta_2 d\theta_1 = 0$$

and since the integrand is periodic in θ_1 and θ_2

$$\int_0^{2\pi}\int_0^{2\pi} \exp\,(j\ell m\theta_2 - j\ell n\theta_1)\,\phi\,(\theta_1,\theta_2)d\theta_2 d\theta_1 = 0$$

and so $\hat{\phi}\,(-\ell n, \ell m) = 0$ for all $\ell \neq 0$, $m > o$ and $n > 0$,

that is,

$$\hat{\phi}(m,n) = 0 \text{ for all } m,n \text{ with } mn < o.$$

The result now follows from Theorem A2.

Q.E.D.

Finally, we note that if f in Theorem A3 is a polynomial, then the converse in Theorem 2 implies that f has polynomial spectral factors. Thus we have the full converse of Theorem 3 for polynomials.

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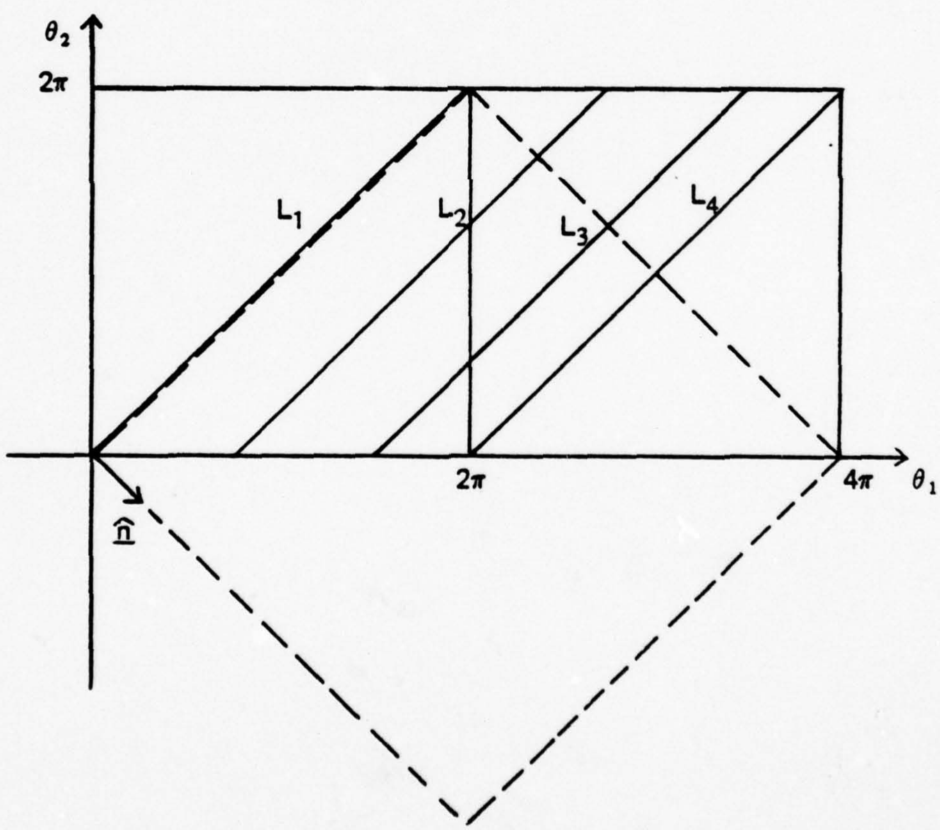

Figure 1.

SYMETRIC HALF-PLANE FILTERS*

J. Murray

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SYMMETRIC HALF-PLANE FILTERS

John Murray

ABSTRACT

A class of two-dimensional recursive digital filters called symmetric half-plane filters is discussed; some properties of these filters are derived and it is shown that in certain situations these properties may give the symmetric half-plane filters both theoretical and practical advantages over previously proposed filters. In particular, they are ideally suited to highly parallel processing.

INTRODUCTION

In the literature on 2-dimensional recursive digital filters, two main types of filter have been studied; these are the quarter-plane filter (e.g.[1], [2]) and the asymmetric half-plane filter [3]. Basically, the two correspond to two different concepts of causality. The general stability conditions for a wide class of filters (including symmetric half-plane) were discussed in [4]; unfortunately, however, those filters are not recursively implementable in general. Here we will consider a class of filters which are recursively implementable, and satisfy the same stability conditions as those in [4].

SYMMETRIC HALF-PLANE FILTERS

By a symmetric half-plane filter we will mean a (causal, recursive) 2-dimensional digital filter, the denominator of whose transfer function is of the form

$$A(Z_1, Z_2) = 1 + \sum_{m=1}^{M} \sum_{n=-N}^{N} a_{mn} Z_1^n \; Z_2^m \qquad (1)$$

This differs from the filters in [4] in that m goes from 1, rather than 0, to M; i.e., this filter omits all of the row m=0 except for the constant term; the asymmetric half-plane filters omit half of this row. The filter (1) is recursively realizable, since the computation of the output at any point depends only on the outputs in previously computed rows; looked at from another point of view, each row of output depends only on previous rows of output. This has two effects; firstly, it focuses attention on the row as the basic element in the filter; secondly, it implies that all the outputs in a given row may be computed in parallel, since each output in a row depends only on outputs in previous rows, and not on any of the outputs of the same row. This is the main practical advantage of this class of filters - it would be of significance, however, only in real-time hardware applications of 2-dimensional filtering, and these seem to be few.

SOME PROPERTIES

Using the methods in [4], one can easily derive the following:

The filter (1) (i.e., the all-pole filter whose denominator is $A(Z_1,Z_2)$) is stable if $A(Z_1,Z_2) \neq 0$ for all (Z_1, Z_2) such that $|Z_1| = 1$ and $|Z_2| \leq 1$.

We note that this set is the same as that for the symmetric half-plane filter in [4]; it is smaller than that for the asymmetric half-plane filters [3]. It is the smallest "instability set" (known to the author) of any recursively implementable class of filters.

However, there is a price to be paid; the amplitude response of the filter is restricted as follows:

If $A(Z_1,Z_2)$ is of the form (1), and if $\frac{1}{A(Z_1,Z_2)}$ is the transfer function of a stable filter, then

$$\int_0^{2\pi} \log \ |A(e^{j\theta_1},e^{j\theta_2})| \ d\theta_2 = 0 \qquad (2)$$

independently of θ_1.

Thus, the average gain along any line of length 2π parallel to the θ_2-axis is constant; or in other words the filter cannot have variations in the θ_1-direction in overall (average) gain. Equivalently, if the cepstrum of $|A(e^{j\theta_1}, e^{j\theta_2})|$ is given by $\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \tilde{a}_{mn} Z_1^n Z_2^m$,

then $\tilde{a}_{on} = 0$, for all n.

This follows immediately from (2) and the definition of the cepstrum; (2) will be proved in a forthcoming paper.

This implies that in order to realize an arbitrary magnitude function, the filter must either have a (nonminimum-phase) numerator polynomial, or the filter must be cascaded with a 1-dimensional filter in Z_1. It is very easy to calculate the ideal amplitude response of this filter.

DESIGN AND IMPLEMENTATION CONSIDERATIONS

It is conceptually convenient (and in a large number of cases, computationally efficient) to implement the convolution in the Z_1-direction by means of the Fourier Transform. (It is assumed from the beginning that the dimension of the array to be filtered is a known fixed constant in the Z_1-direction, i.e., each row is of the same fixed width). From this point of view, and regarding each row as a single entity described by its 1-dimensional Z-transform, the coefficients a_{mn} in (1) are irrelevant; what matters are the M functions

$$\hat{a}_m(e^{j\theta_1}) = \sum_{n=-N}^{N} a_{mn}(e^{j\theta_1})^N.$$

Further, the stability requirement for the filter is equivalent to the requirement that for each fixed θ_1, the filter defined by

$$A_{\theta_1}(Z_2) = \frac{1}{\sum_{m=1}^{M} \hat{a}_m(e^{j\theta_1}) Z_2^m + 1}$$

is 1-dimensionally stable. Finally, the functions $\hat{a}_m(e^{j\theta_1})$ do not have to be

analytic or meromorphic functions; this is seen by letting $N \to \infty$. In other words, the roots of $\hat{a}_m(e^{j\theta_1})$ can vary quite arbitrarily with θ_1. Thus, we can design the one-variable filter $A_{\theta_1}(Z_2)$ by any one of the usual one-variable design methods we choose (yielding a stable filter) for each θ_1: the result will be a stable two-variable filter: further, if our one-variable method gives poles and zeros explicitly, we have the same for our two variable filter, which can therefore be expressed as a cascade of filters of degree 1 in Z_2. Finally, if one desires a filter of finite degree in Z_1, one can solve the following approximation problem (for each m, $1 \le m \le M$); minimize (over b_{mn})

$$\left\| \sum_{n=-N}^{N} b_{mn} e^{jn\theta_1} - \hat{b}_m(e^{j\theta_1}) \right\| \text{ subject to}$$

$$\left| \sum_{n=-N}^{N} b_{mn} e^{jn\theta_1} \right| < 1 \text{ for all } \theta_1, \text{ where}$$

$\hat{b}_m(e^{j\theta_1})$ denotes the m-th pole of $A_{\theta_1}(Z_2)$ as a function of θ_1, and $\| \; \|$ denotes some error norm.

Hopefully this will become clearer on consideration of the following example.

EXAMPLE

We wish to design a filter with second-order Butterworth response in Z_2 to approximate the fan filter whose passband is the set $|\theta_2| < |\theta_1|$. For fixed θ_1, therefore, the filter is a 1-dimensional filter whose passband is the set $|\theta| < |\theta_1|$. Using the bilinear transform technique, we find the second-order continuous Butterworth filter

$$\frac{1}{1 + \sqrt{2}\, s/\omega_c + s^2/\omega_c^2}$$

transforms into

$$\frac{\omega_c^2 (1 + Z_2)^2}{\omega_c^2 - \sqrt{2}\, \omega_c + 1 + (2\omega_c^2 - 2) Z_2 + (\omega_c^2 + \sqrt{2}\, \omega_c + 1) Z_2^2} \tag{3}$$

while its (stable) poles transform into

$$\{1 + \omega_c(\tfrac{-1+j}{\sqrt{2}})\}/\{1 - \omega_c(\tfrac{-1+j}{\sqrt{2}})\}$$

and

$$\{1 + \omega_c(\tfrac{-1-j}{\sqrt{2}})\}/\}1 - \omega_c(\tfrac{-1-j}{\sqrt{2}})\}$$

for $\omega_c \geq 0$.

In accordance with the usual frequency warping, we take $\omega_c = |\tan \theta_1/2|$; however, we note that this causes stability problems at $\theta_1=0$ and $\theta_1=\pi$; we therefore take a small perturbation of ω_c, e.g.,

$$\omega_c = \sqrt{\frac{(\sin \theta_1/2)^2 + \epsilon}{(\cos \theta_1/2)^2 + \epsilon}} \tag{4}$$

The filter can now be directly implemented by multiplying the Fourier transform of the previous output rows by the appropriate functions according to (3) and (4) and performing the recursion from row to row directly.

Alternatively, if a finite-order (in Z_1) filter is desired, we must solve the problem: Minimize (over b_{1n})

$$\int_0^{2\pi} \left| \frac{\sqrt{2}+(j-1)|\tan \theta_1/2|}{\sqrt{2}+(1-j)|\tan \theta_1/2|} - \sum_{n=-N}^{N} b_{1n} e^{jn\theta_1} \right| d\theta_1$$

subject to $\left| \sum_{n=-N}^{N} b_{1n} e^{jn\theta_1} \right| < 1$ for all θ_1

and similarly for the other root. The resulting filter may then be implemented in

cascade. It should be pointed out that while the above optimization problem is not simple, it is one-dimensional.

Finally, the above example was chosen for simplicity and convenience rather than realism. Clearly a Butterworth filter is not optimal for this problem, especially when it yields a design which is not all-pole; if we have to store input rows (as well as output rows) we may as well use them, and design an elliptic type filter; alternatively, we might use a filter which is all-pole in its discrete form.

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THE ENCIRCLEMENT CONDITION:

AN APPROACH USING ALGEBRAIC TOPOLOGY*

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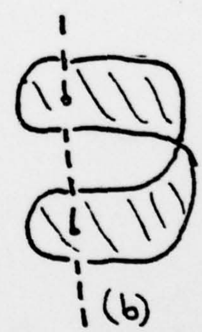
Introduction

The concepts delineated in this paper arose in part from an introductory study of Riemann surfaces. Associated with an analytic function is a Riemann surface. It has the property that the image of simply connected regions in the complex plane are simply connected on the Riemann surface.

The point made here is that the Nyquist criterion is trivial for simply connected regions. Moreover, if one can work on the Riemann surface, this trivality carries over to the general case. To illustrate the point, let Figure 1-a be the image of the right half plane under an analytic map. The region is not simply connected. Figure 1-b shows the "same region" as it might appear on an appropriate Riemann surface. Here the region is simply connected.

(a)



(b)

Figure 1

Under the hypothesis that f is bounded at infinity, the boundary of the region in Figure 1-b is the image of the imaginary axis as indicated by the darkened line in Figure 1-a.

Now remove "-1" (this may be a set of points) from the Riemann surface. The essential argument we need is that the Nyquist contour in the complex plane is homotopic to zero if and only if "-1" is in the interior of its image on the Rimann surface.

Although motivated by the intrinsic properties of Riemann surface, this paper drops any further discussion of the concept so as to simplify the exposition. Instead, the paper exploits the fact that the Nyquist contour is a simple closed curve in the complex plane. Mathematically we draw only on the intuitive concept of homotopic triviality as found in algebraic topology.

In the sequel, we prove the classical stability results via homotopy theory. In particular, we utilize covering space theory. We believe our analysis is clearer and more intuitive than has hitherto appeared. Moreover, we believe that this research indicates that the nub of the Nyquist criteria is in fact homotopy theory. In a future paper, we will generalize these results to functions of several complex variables and their application to the stability of of multi-dimensional digital filters.

Mathematical Preliminaries

Firstly, let $\mathbb{C}$, $\mathbb{C}_+$, and $\overset{\circ}{\mathbb{C}}_+$ be the complex plane, the closed right half plane, and the open right half plane respectively. Let $\overset{\circ}{\mathbb{C}}_- = \mathbb{C} - \mathbb{C}_+$. Basic to homotopy theory is the concept of a path. A path or a curve in the complex plane is a continuous function $\gamma:[0,1]\to\mathbb{C}$. For this paper all paths are understood to be

rectifiable--i.e. γ is also a function of bounded variation. (2) γ is a closed path if $\gamma(0) = \gamma(1)$. γ is a simple closed path if γ is a closed path and has no self intersections. The image of $I = [0,1]$ under γ is called the trace of γ and is denoted by $\{\gamma\}$.

Two closed curves γ_0 and γ_1 are homotopic ($\gamma_0 \sim \gamma_1$) in $\mathbb{C}$ if there exists a continuous function Γ: $I \times I \to \mathbb{C}$ such that:

(a) $\Gamma(s,0) = \gamma_0(s) \qquad 0 \leq s \leq 1$

(b) $\Gamma(s,1) = \gamma_1(s) \qquad 0 \leq s \leq 1$

(c) $\Gamma(0,t) = \Gamma(1,t) \qquad 0 \leq t \leq 1$

Intuitively, γ_0 is homotopic to γ_1 if one can continuously deform γ_0 into γ_1. Moreover, it is easily shown that the homotopy relation is an equivalence relation. (4) (5)

Another important property of a closed curve is its index. The index of a closed curve, γ, with respect to a point "a" not in $\{\gamma\}$ is:

$$n(\gamma;a) = \frac{1}{2\pi i} \int_{\gamma} (z-a)^{-1} dz \qquad (2)$$

Observe that

$$\int_{\gamma} (z-a)^{-1} dz = \int_{\gamma} d(\ln(z-a)) = \int_{\gamma} d(\ln|(z-a)|) + i\int_{\gamma} d(\arg(z-a))$$

$$= i\int_{\gamma} d(\arg(z-a))$$

This integral therefore measures i times the net increase in angle that the ray r of Figure 3 accumulates as its tip traverses the curve γ.

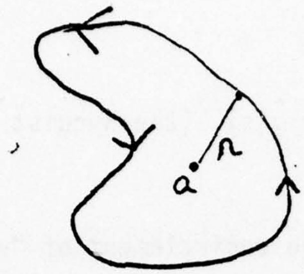


Figure 2

Following the comments of J. Barman and J. Katzenelson (1), for the integral to be well defined it is necessary to specify the appropriate branch of arg(z-a) at each point of the integration. We will assume the choice of branch as outlined in (1).

Finally, we point out that this definition of index (encirclement) is a special case (i.e. in the plane) of the general topological concept of Brouwer degree. (4) (5) (6)

At any rate $n(\gamma;a) = 0$ if and only if γ is homotopic to a point in $\mathbb{C} - \{a\}$. (cf. prop 5.4, ref. 2) Simply then, a closed curve γ does not encircle the point "-1" if and only if γ is homotopic to a point in $\mathbb{C} - \{-1\}$. We will henceforth refer to such a γ as being homotopically trivial.

Conversely, γ encircles "-1" if and only if γ cannot be continuously deformed to a point in $\mathbb{C} - \{-1\}$. Clearly these ideas indicate that the Nyquist encirclement condition is fundamentally a homotopy concept.

To further illucidate the point, let $\hat{g}(s)$ be a rational transfer function depicting the open loop gain of a scalar single loop feedback system. Suppose all poles of $\hat{g}(s)$ are in $\mathring{\mathbb{C}}_-$ and $\hat{g}(\infty) \leq M < \infty$. Via the Nyquist Criteria, the closed loop system is stable if and only if $\hat{h}(s) = \hat{g}(s)/(1+\hat{g}(s))$ is stable; if and only if the image of the

imaginary axis, under $\hat{g}(s)$, (the Nyquist plot of $\hat{g}(s)$) does not pass nor encircle "-1".

Specifically, the encirclement of "-1" by the Nyquist plot implies there exists at least one s_0 in $\mathring{\mathbb{C}}_+$ such that $\hat{g}(s_0) = -1$. Thus the Nyquist contour is homotopically trivial in $\mathbb{C}_+ - \{\hat{g}^{-1}(-1)\}$ if and only if the Nyquist plot is homotopically trivial in $\hat{g}(\mathbb{C}_+)-\{-1\}$.

Motivation for this approach also arose from a close scrutiny of the classical proof of the Nyquist criteria which depends on the argument principle. The argument principle supplies unnecessary although specific information in that it counts the number of times "-1" is encircled. This may account for the apparent difficulty in *generalizing the Nyquist criteria*. Nevertheless, the affinity between homotopy and encirclement ideas led the authors to a minor study of algebraic topology.

In our setting, algebraic topology establishes a topologically invariant relationship between a metric space, X, and an algebraic group called the fundamental group of X, denoted by $\pi(X)$. The relationship is topologically invariant in that homeomorphic spaces have isomorphic fundamental groups.

Specifically, the fundamental group is a set of equivalence classes of closed curves. Each equivalence class consists of a set of curves homotopically equivalent. The group operation is "concatenation" of curves.

For example, the fundamental group of $\mathcal{C}$ consists of one element, $i_{\mathcal{C}}$, the identity, since all closed curves are homotopic to zero. If $X = \mathcal{C} - \{-1\}$, then $\pi(X)$ has a countable number of elements: i_X (the identity) equal to the equivalence class of all closed curves not encircling "-1" and the remaining elements, μ_n $(n = 1, 2, 3 \ldots)$ consisting of the equivalence class of all closed curves encircling "-1", n times. Moreover, μ_i concatenated with μ_k is equal to the element μ_{k+i}.

Now let X and Y be metric spaces. Let $f\colon X \to Y$ be locally homeomorphic. In particular, assume that for each point y in Y there exists an open neighborhood G of y such that each connected component of $f^{-1}(G)$ is homeomorphic to G under the map f. Under this condition X is said to be a covering space of Y. (2) (4) Also let $\pi(X)$ and $\pi(Y)$ be the fundamental groups associated with X and Y respectively. With these assumptions, f effects a group isomorphism (i.e. a one to one onto mapping preservinggroup operations) ϕ_f between $\pi(X)$ and a subgroup of $\pi(Y)$ as in the following diagram. (4) (5)

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "f"'] & \pi(X) \arrow[d, "\varphi_f"] \\ Y \arrow[r, "F"'] & N \subset \pi(Y) \end{tikzcd}$$

Figure 3

F is the functor which establishes the relationship between a topological space and its fundamental group.

Before judiciously tailoring the complex plane so as to apply the above result, we distinguish between a critical point and a critical value. A point z_0 in $\mathbb{C}$ is a critical point of a differentiable function f if $f'(z_0) = 0$. A critical value of f is any point $w = f(z_0)$ whenever z_0 is a critical point.

Now suppose $f:\mathbb{C} \to \mathbb{C}$ is a rational function whose set of poles is $P = \{P_1, \ldots, P_n\}$. Let $Q = \{q_1, \ldots, q_m\}$ be the set of all points in $\mathbb{C}$ such that $f(q_i)$ is a critical value of f. Note that there may be q_i's which are not critical points. To see this consider $g(z) = z^2(z-a)$. $g(0) = 0$ implies "0" is a critical value of g, but $g(a) = 0$ with $g'(a) \neq 0$. Finally, define $T = \{t_i | t_i = f^{-1}(-1), i=1,\ldots,n\}$. Note also that since f is a rational function, P, Q and T are finite sets. Define $X = \mathbb{C} - \{P\ Q\ T\}$ and define $Y = f(X)$.

<u>Lemma 1</u>: Under the above hypothesis, X is a covering space of Y.

Proof: For X to be a covering space of Y, each y in Y must have an open neighborhood G_Y such that each component of $f^{-1}(G_Y)$ is homeomorphic to G_Y. Using the inverse function theorem (7) we construct such a neighborhood.

Let $\{x_1, \ldots, x_k\} = f^{-1}(y)$ where again the finiteness of this set is a consequence of the rationality of f. Let $W_1, \ldots, W_k$ be disjoint open neighborhoods of $x_1, \ldots, x_k$ respectively. Since

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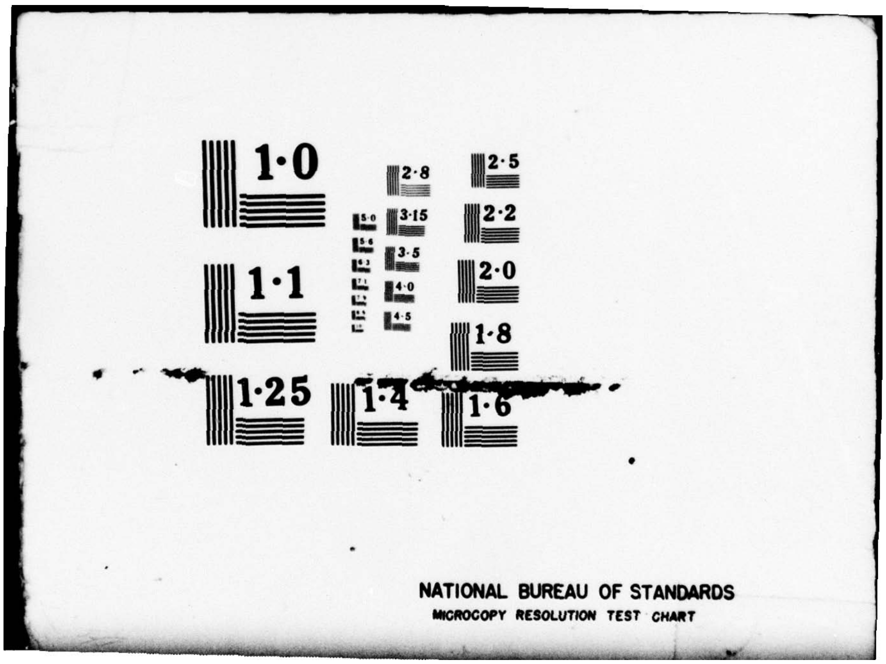
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f is analytic on X and since $f'(x) \neq 0$ for all x in X, the inverse function theorem guarantees that there exist open neighborhoods $U_i \subset W_i$ $(i = 1, \ldots, k)$ such that U_i is homomorphic to $V_i = f(U_i)$, where it follows that V_i is an open neighborhood of y.

Thus $f^{-1}(V_1 \cup \ldots \cup V_k) = U_1 \cup \ldots \cup U_n$. Define $V = V_1 \cap \ldots \cap V_k$. Clearly V is an open neighborhood of y and $f^{-1}(V) \subset U_1 \cup \ldots \cup U_k$. Since each U_i is homeomorphic to $V_i \supset V$, $f^{-1}(V) \subset U_i$ is homeomorphic to V.

Therefore each y in Y has an open neighborhood G_y such that $f^{-1}(G_y)$ has each of its components homeomorphic to G_y. It follows that X is a covering space of Y.

Corollary: The fundamental group $\pi(X)$ of X is *isomorphic to a sub-* broup N of $\pi(Y)$.

This corollary says that a closed curve in X is homotopically trivial if and only if its image under f is homotopically trivial.

The Scalar Nyquist Criterion

In this section we apply the above corollary to an "ugly" Nyquist contour. After proving the Nyquist Theorem using this "ugly" contour we relate it to the usual Nyquist contour. This will establish the classical result.

Let $\hat{g}(s)$ be a rational function which represents the open loop gain of a scalar, single-loop unity feedback system. We assume $\hat{g}(s) \neq 0$. Thus the closed loop system has a transfer function

$\hat{h}(s) = \hat{g}(s)(1+\hat{g}(s))^{-1}$.

We will say that the closed loop system $\hat{h}(s)$ is stable if and only if $\hat{h}(s)$ has all its poles in $\mathbb{C}_-$ and $h(\infty) < \infty$.

Let $P = \{p_1, \ldots, p_n\}$ be the set of poles of $\hat{g}(s)$ and let $Q = \{q_1, \ldots, q_m\}$ be the set of points q_i such that $\hat{g}(q_i)$ is a critical value of $\hat{g}$. Define $T = \{t_i | t_i = f^{-1}(-1), i = 1, \ldots, l\}$. Finally, let $X = \mathbb{C} - \{P \cup Q \cup T\}$ and let $Y = f(X)$. Lemma 1 implies X is a covering space of Y under the mapping f.

Assume for the present that $\hat{g}(i\omega) \neq$ "-1" for $-\infty \leq \omega \leq \infty$. The first task is to construct the "ugly" Nyquist contour as well as the classical contour. Define the ulgy contour to be γ_R where $\gamma_R: I \to X \subset \mathbb{C}$ is a path whose trace is illustrated in Figure 4-2. Note that R is chosen strictly greater than max $(|p_i|, |q_j|, |t_k|)$ for $1 \leq i \leq n$, $1 \leq j \leq m$, and $1 \leq k \leq l$. The indentations, along the imaginary axis into the right half plane, are of radius $0 < \varepsilon < \varepsilon_0$. These semicircular indentations are made around all points of P lying on the imaginary axis and around all points q_i of Q lying on the $i\omega$-axis with $q_i \notin T$. The other "indentations" (again of radius ε, $0 < \varepsilon < \varepsilon_0$) are slits into $\mathbb{C}_+$ which encircle all points of P and all points q_i of Q ($q_i \notin T$) which are in $\mathring{\mathbb{C}}_+$ so as to eliminate these points from the interior of the contour. We have also labeled these slits $\mu_1, \ldots, \mu_r$ where each μ_i maps an appropriate subinterval of I onto the specified subset of $\{\gamma_R\}$. The parellel lines, connecting a pole in $\mathring{\mathbb{C}}_+$ with the semicircular portion of γ_R, are actually the same line segment (slit) traversed

in opposite directions. Note that we have indicated the usual counterclockwise orientation to the path. Thus the only points encircled by γ_R are points of T which are in $\mathring{\mathbb{C}}_+$.

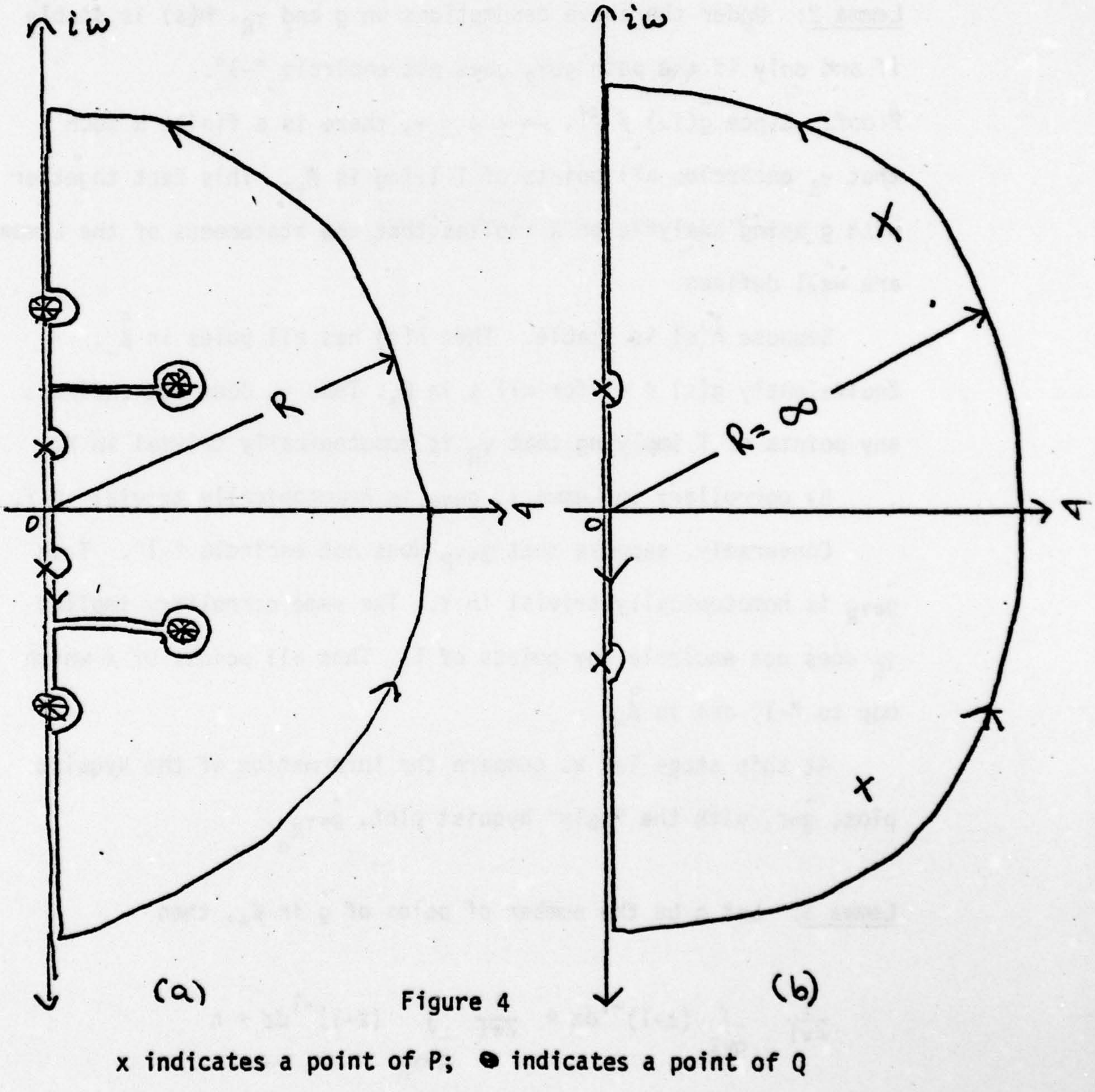

Figure 4

x indicates a point of P; ⊛ indicates a point of Q

Let Γ denote the classical Nyquist contour where $\Gamma: I \to \mathbb{C} \cup \{\infty\}$ as indicated in Figure 4-b.

<u>Lemma 2</u>: Under the above assumptions on $\hat{g}$ and γ_R, $\hat{h}(s)$ is stable if and only if the path $\hat{g}\circ\gamma_R$ does not encircle "-1".

Proof: Since $\hat{g}(i\omega) \neq -1$, $-\infty \leq \omega \leq \infty$, there is a finite R such that γ_R encircles all points of T lying in $\mathbb{C}_+$. This fact together with g being analytic on X implies that the statements of the Lemma are well defined.

Suppose $\hat{h}(s)$ is stable. Then $\hat{h}(s)$ has all poles in $\mathring{\mathbb{C}}_-$. Equivalently $\hat{g}(s) \neq -1$ for all s in $\mathbb{C}_+$. Thus γ_R does not encircle any points of T implying that γ_R is homotopically trivial in X.

By corrollary to Lemma 1, $\hat{g}\circ\gamma_R$ is homotopically trivial in Y.

Conversely, suppose that $\hat{g}\circ\gamma_R$ does not encircle "-1". Then $\hat{g}\circ\gamma_R$ is homotopically trivial in Y. The same corrollary implies γ_R does not encircle any points of T. Thus all points of X which map to "-1" are in $\mathring{\mathbb{C}}_-$.

At this stage let us compare the information of the Nyquist plot, $\hat{g}\circ\Gamma$, with the "ugly" Nyquist plot, $\hat{g}\circ\gamma_{R_0}$.

<u>Lemma 3</u>: Let n be the number of poles of $\hat{g}$ in $\mathbb{C}_+$, then

$$\frac{1}{2\pi i} \int_{\hat{g}\circ\Gamma} (z-1)^{-1} dz = \frac{1}{2\pi i} \int_{\hat{g}\circ\gamma_R} (z-1)^{-1} dz + n$$

Proof: Consider that

$$\frac{1}{2\pi i}\int_{\hat{g}\circ\gamma_R}(z-1)^{-1}dz = \frac{1}{2\pi i}\int_{\hat{g}\circ\Gamma}(z-1)^{-1}dz + \sum_{k=1}^{r}\int_{\hat{g}\circ\mu_k}(z-1)^{-1}dz$$

But $\int_{\hat{g}\circ\mu_k}(z-1)^{-1}dz = \int_{\mu_k}(\hat{g}(z)-1)^{-1}\hat{g}'(z)dz$

If μ_k encircles a point of Q, then $(\hat{g}(z)-1)^{-1}\hat{g}'(z)$ is analytic in the region bounded by μ_k and thus the integral approaches zew uniformly for arbitrarily small ε. Thus the integral is zero these points.

If μ_k encircles a pole of $\hat{g}(s)$ then since $(\hat{g}(z)-1)^{-1}\hat{g}'(z)$ is analytic in the region bounded by μ_k:

$$\int_{\mu_k}(\hat{g}(z)-1\hat{g}'(z)dz = \int_{\mu_k}d(\ln(\hat{g}(z)-1) = \ln(\hat{g}(z)-1)\Big|_{\mu_k} = 2\pi i$$

for a suitable branch of the logarithm. The integral comes out as negative $2\pi i$ since μ_k was traversed in the clockwise direction. The conclusion of the lemma now follows.

At this point let us remove the restriction that $\hat{g}(i\omega) \neq -1$ for $-\infty \leq \omega \leq \infty$. We now give a proof of the classical Nyquist criterion using the above concepts.

<u>Theorem 1</u>: Let $\hat{g}(s)$ be as above with the earlier restriction removed. Then $\hat{h}(s)$ is stable if and only if the Nyquist plot of $\hat{g}(s)$ does not pass through "-1" and encircles "-1" exactly n times

where n is the number of poles of $\hat{g}(s)$ in $\mathring{\mathbb{C}}_+$.
Proof: Suppose $\hat{h}(s)$ is stable, then all poles of $\hat{h}(s)$ are in $\mathbb{C}_-$ and $\hat{h}(\infty) < \infty$. Thus $\hat{g}(\infty) \neq -1$ and $\hat{g}(s) \neq -1$ for all s in $\mathbb{C}_+$. Therefore via Lemmas 2 and 3 the Nyquist plot encircles "-1" exactly n times.

Conversely suppose the Nyquist plot encircles "-1" exactly n times and does not pass through "-1". Thus $\hat{g}(\infty) \neq -1$ which implies $\hat{h}(\infty) < \infty$. Moreover since $\hat{g}\circ\Gamma$ encircles "-1" n times and there are n poles of $\hat{g}(s)$ in $\mathring{\mathbb{C}}_+$, we know that $\hat{g}\circ\gamma_R$ is homotopically trivial. Thus γ_R is homotopically trivial. Consequently there are no points t_i in $\mathbb{C}_+$ such that $\hat{g}(t_i) = -1$. Thus $\hat{h}(s)$ is stable.

Matrix Case

Let the entries of an nxn matrix $\hat{G}(s)$ be rational functions in the complex variable s. Suppose $\hat{G}(s)$ depicts the open loop gain of the single loop feedback system of Figure 6.

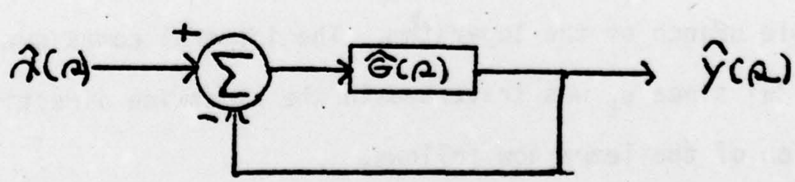

Figure 5

$\hat{x}(s)$ and $\hat{y}(s)$ are n vectors whose entries are also rational functions of s which represent the input and output of the system respectively.

In this article, we assume each entry of $\hat{G}(s)$ is bounded at $x = \infty$. Thus $\hat{G}(s)$ as a mapping, $\hat{G}(\cdot):\mathbb{C} \to \mathbb{C}^{nxn}$, is analytic on $\mathbb{C}$ except at a finite number of points, the poles of its entries.

For Figure 6 to be well defined we require that det $[I+\hat{G}(s)] \neq 0$ Thus there exists a closed loop convolution operator, H, such that $y = H*x$. Moreover the Laplace transform of H, $\hat{H}(s)$ satisfies

$$\hat{H}(s) = \hat{G}(s)[I+\hat{G}(s)]^{-1}$$

For the system of Figure 6 to be stable, $\hat{H}(s)$ must have all its poles in $\mathbb{C}_-$ and have all its entries bounded at $s = \infty$.

Under the assumptions on $\hat{G}(s)$, the following factorization is valid:

$$\hat{G}(s) = N(s)D^{-1}(s)$$

where N(s) and D(s) are right co-prime, polynomial matrices in s with $\det[D(s)] \neq 0$. Moreover s_0 is a pole of $\hat{G}(s)$ if and only if it is a zero of det[D(s)]. (9)

Desoer and Schulman (3) have shown that the close loop operator H is stable if and only if $\det[N(s)+D(s)] \neq 0$ for s in $\mathbb{C}_+$ and $\det[I+G(\infty)] \neq 0$. Using this fact, we state and prove the following:

<u>Theorem 2</u>: H is stable if and only if (1) the Nyquist plot of det[N(s)+D(s)] does not encircle nor pass through "0", and (2) $\det[I+\hat{G}(\infty)] \neq 0$.

Proof: By hypothesis we require $\det[I+G(\infty)] \neq 0$. Therefore we only must verify that $\det[N(s)+D(s)] \neq 0$ for $\mathrm{Re}(s) \geq 0$ if and only if the Nyquist plot of $\det[N(s)+D(s)]$ does not pass through nor encircle "0".

Now the Nyquist plot of $\det[N(s)+D(s)]$ passes through "0" if and only if $\det[N(s)+D(s)]$ has a zero on the imaginary axis--i.e. if and only if the closed loop system has a pole on the imaginary axis.

Finally assume the Nyquist plot of $\det[N(s)+D(s)]$ does not pass through "0". Observe that $\det[N(s)+D(s)]$ is a polynomial and thus a rational function. As per Theorem 1, appropriately define X and Y so that X is a covering space of Y under the map $\det[N(\cdot)+D(\cdot)]$. The above lemmas imply that the Nyquist plot of $\det[N(s)+D(s)]$ is homotopically trivial if and only if there exists a point s_0 in $\overset{o}{\mathbb{C}}_+$ such that $\det[N(s)+D(s)] = 0$. The assertion of the theorem now follows.

Observe that if one assumes the open loop gain to be stable (i.e. $\hat{G}(s)$ has all poles in $\overset{o}{\mathbb{C}}_+$) then $\det[I+\hat{G}(s)]$ can replace $\det[N(s)+D(s)]$ in the above theorem. This follows since for all s in $\mathbb{C}_+$, $\det[N(s)+D(s)] = \det[I+\hat{G}(s)] \det[D(s)]$ with $\det[D(s)] \neq 0$. Thus in $\mathbb{C}_+$ $\det[N(s)+D(s)]$ has a zero if and only if $\det[I+\hat{G}(s)]$ has a zero.

Finally, it is worthwhile to point out the relationship between the above formulated multivariable Nyquist criterion and that formulated by Barman and Katznelson[!] For this purpose we let $\lambda_j(iw)$; $j=1, \ldots, n$; denote the n eigen values of $\hat{G}(iw)$. In general parameterization of these function by iw is not uniquely determined but one can always formulate such a function. Moreover these functions are piecewise

analytic and can be concatonated together in such a way as to form a closed curve which Barman and Katznelson term the Nyquist plot of $\hat{G}(s)$. Now, since

$$\det[I + \hat{G}(iw)] = \prod_{j=1}^{n} [1 + \lambda_j (iw)]$$

and the degree of a product is the sum of the degrees of the individual factors and also equals the degree of the cancatonation of the factors, the degree of the Barman and Katznelson plot with respect to "-1" coincides with the degree of our plot with respect to "0". As such, even though the two plots are different their degrees coincide and hence either can be used for a stability test.

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CONTINUATION METHODS FOR STABILITY ANALYSIS OF MULTIVARIABLE FEEDBACK SYSTEMS*

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Continuation Methods for Stability Analysis of Multivariable Feedback Systems*

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Abstract

Techniques for implementation of a Nyquist stability result for a linear time invariant multivariable feedback system are described. The approach is based on continuation methods for computing the system's eigenvalue loci.

I. INTRODUCTION

The classical Nyquist stability criterion for single-input single-output, linear time-invariant feedback systems has only recently been generalized to multivariable feedback systems [1,2]. Stability theorems are expressed in terms of the eigenvalue loci of the open loop transfer function G(s) of the system. In particular if G(s) is stable, i.e., G(s) has no poles in the right half of the s-plane or on the $j\omega$-axis, then a linear time-invariant multivariable feedback system with n inputs and n outputs is stable if and only if its generalized Nyquist plots(union of eigenvalue loci) does not pass through or encircle the (-1, 0) point [1]. In order to apply the multivariable Nyquist criterion, it is thus necessary to compute the eigenvalue loci as a function of frequency. For a given frequency, the eigenvalues can be calculated by using classical techniques. Since the eigenvalues are functions of frequency, normally one would have to repeat the entire computational procedure for each frequency. In the actual stability analysis, this repetition is however, impractical. Our approach to the stability analysis of multivariable feedback systems is based on continuation methods. The basic idea of all continuation methods is to convert the solution of a parameterized family of algebraic problems into the solution of a differential equation. Then if one can find the solution

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of an initial problem by using classical methods the solutions to the other problems can be obtained by integrating the associated differential equation with the initial solution as an initial condition.

II. EIGENVECTOR APPROACH

Our first method is based on the approach described by Faddeev and Fadeeva [3] and Van Ness et. al. [4]. A differential equation is written with the eigenvalues as dependent variables and the frequency as variable parameter. We then compute a set of initial eigenvalues by classical analysis techniques and integrate the resulted differential equation to obtain the required eigenvalues for each frequency. The eigenvalues $\lambda_i(\omega)$ of $G(j\omega)$ and their complex conjugates $\bar{\lambda}_i(\omega)$ satisfy

$$G(j\omega)X_i(\omega) = \lambda_i(\omega)X_i(\omega) \quad i=1,2,\ldots,n \tag{1}$$

and

$$G^*(j\omega)V_i(\omega) = \bar{\lambda}_i(\omega)V_i(\omega) \quad i-1,2,\ldots,n \tag{2}$$

where $X_i(\omega)$ and $V_i(\omega)$ are the corresponding eigenvectors of $\lambda_i(\omega)$ and $\bar{\lambda}_i(\omega)$ respectively, and $G^*(j\omega)$ is the complex conjugate transpose matrix of $G(j\omega)$.

We differentiate (1) with respect to ω to yield

$$\frac{d\lambda_i}{d\omega} = \frac{<\frac{dG}{d\omega}\, X_i \quad V_i>}{<X_i,\, V_i>}, \quad i = 1,\, 2,\, \ldots,\, n. \tag{3}$$

The differential equations involving X_i and V_i are obtained as

$$\frac{dX_i}{d\omega} = \sum_{j=1}^{n} \alpha_{ij}X_j, \quad i= 1,\, 2,\, \ldots,\, n. \tag{4}$$

$$\frac{dV_i}{d\omega} = \sum_{j=1}^{n} \beta_{ij}V_j, \quad i = 1,\, 2,\, \ldots,\, n. \tag{5}$$

where

$$\alpha_{ii}=0, \quad \alpha_{ij}= \frac{<\frac{dG}{d\omega}\, X_i,\, V_j>}{(\lambda_i-\lambda_j)<X_j,V_j>} \quad i \neq j. \tag{6}$$

$$\beta_{ii} = 0, \quad \beta_{ij} = \frac{<\frac{dV_i}{d\omega},\, X_j>}{<V_j,\, X_j>} \quad i \neq j. \tag{7}$$

Starting with a set of predetermined initial conditions $\lambda_i(0) = \lambda_{io}$, $X_i(0) = X_{io}$ and $V_i(0) = V_{io}$ for $i = 1, 2, \ldots, n$, we integrate (3), (4)

and (5) to obtain the required eignevalues for each frequency. The eigenvalue loci are computed in a continuous manner by numerical integration.

III. JACOBIAN METHOD

For an nth order system, the above algorithm requires the numerical integration of a set of 3n equations and the computation of two sets of unwanted variables--namely the eigenvectors X_i and V_i. These disadvantage, can easily be avoided if the characteristic equation for the multivariable feedback system can be predetermined. A much simpler method can be formulated based on the approach for finding multiple solutions for a nonlinear equation developed by Chao et. al. [5].

Let the characteristic equation of $G(j\omega)$ be given by an nth order polynomial in eigenvalue λ with complex coefficients

$$f[\lambda(\omega)] = |\lambda I - G(j\omega)| = 0. \tag{8}$$

Instead of solving (8) directly for each frequency, we consider two simultaneous differential equations of the form

$$\begin{aligned} \frac{df}{dt} &= -f(t) \qquad f(0) = f[\lambda(\omega_0)] = 0 \\ \frac{d\omega}{dt} &= \pm 1 \qquad \omega(0) = \omega_0. \end{aligned} \tag{9}$$

Assuming the nonsingularity of the Jacobian Matrix

$$J = \begin{bmatrix} \frac{\partial f}{\partial \lambda} & \frac{\partial f}{\partial \omega} \\ \frac{\partial \omega}{\partial \lambda} & \frac{\partial \omega}{\partial \omega} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial \lambda} & \frac{\partial f}{\partial \omega} \\ 0 & 1 \end{bmatrix}, \tag{10}$$

in the x-ω space the algorithm (9) reduces to

$$\begin{bmatrix} \frac{d\lambda}{dt} \\ \frac{d\omega}{dt} \end{bmatrix} = J^{-1} \begin{bmatrix} -f \\ \pm 1 \end{bmatrix} ; \quad \begin{bmatrix} f(0) \\ \omega(0) \end{bmatrix} = \begin{bmatrix} 0 \\ \omega_0 \end{bmatrix}. \tag{11}$$

It is seen from the solution of (9)

$$\begin{aligned} f(t) &= 0e^{-t} \equiv 0 \\ \omega &= \pm t. \end{aligned} \tag{12}$$

that for any admissible pair of ω_0 and $\lambda(\omega_0)$ satisfying (8), the corresponding trajectory will remain on the solution curve f=0 as ω changes. The + or - sign is chosen depending on whether one would like to increase or decrease ω. Equation (11) may now be solved by any numerical integration techniques and the eigenvalue loci can be traced automatically by integrating only a second order differential system.

IV. EXAMPLE

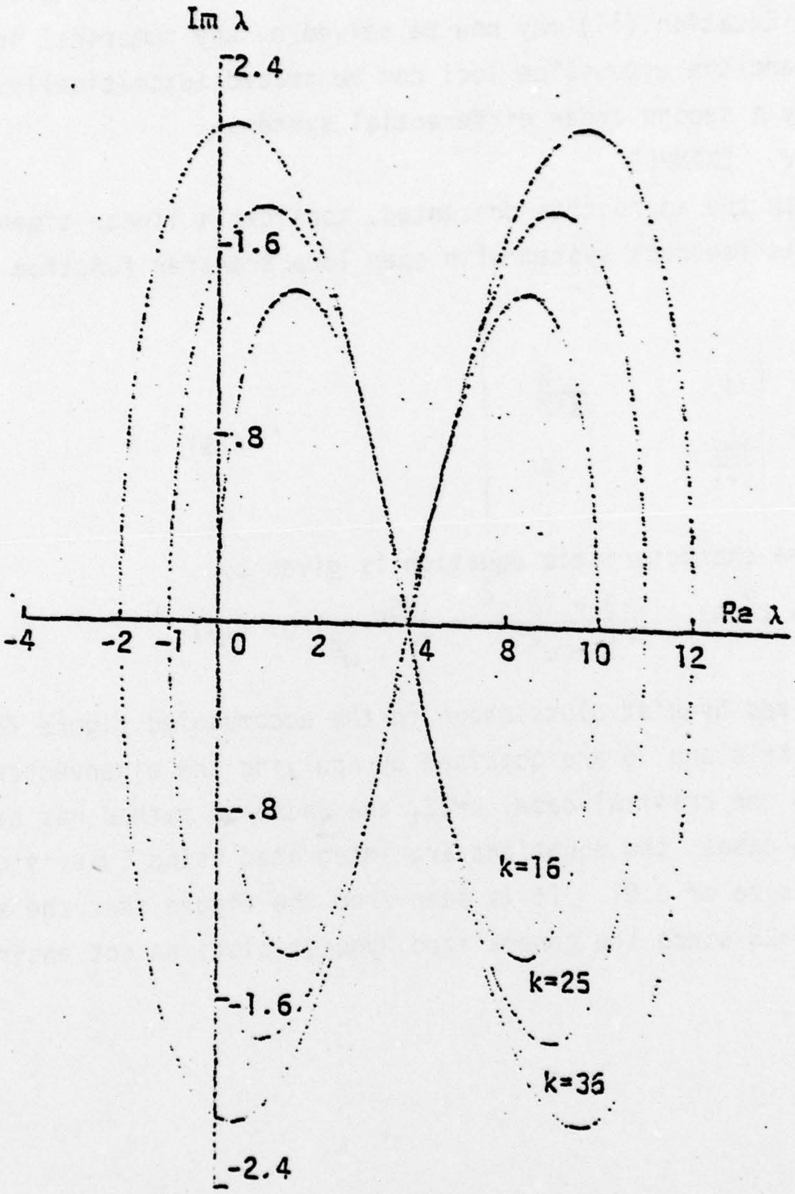
To illustrate the approaches presented, consider a linear time-invariant, multivariable feedback system with open loop transfer function characterized by

$$G(s)=\begin{vmatrix} 4 & \frac{k}{s+2} \\ \frac{s+2}{s+1} & 4 \end{vmatrix} \tag{13}$$

for which the characteristic equation is given by

$$f[\lambda(\omega)] = \lambda^2-8\lambda + \left(\frac{-9 + 16\,\omega^2}{1 + \omega^2} + j\frac{25\omega}{1 + \omega^2}\right)=0. \tag{14}$$

The generalized Nyquist plots shown in the accompanied figure for the cases where k=16 and 36 are obtained by applying the eigenvector approach where as in the critical case, k=25, the Jacobian method has been used. In all three cases, the equations are integrated using Euler's method with a step size of 0.01. It is seen from the figure that the system is stable for k<25 since the generalized Nyquist plots do not encircle -1 point.



FIGURE

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